Problems and Solutions

This section publishes problems and solutions proposed by readers and editors. Readers are welcome to submit solutions to the following problems. Your solutions, if chosen, will be published in the next issue, bearing your full name and address. A publishable solution must be correct and complete, and presented in a well-organised manner. Moreover, elegant, clear and concise solutions are preferred. Solutions should reach the editors before 31 May, 1991.

Readers are also invited to propose problems for future issues. Problems should be submitted with solutions, if any. Relevant references should be stated. Indicate with an ° if the problem is original and with an * if its solution is not available.

All correspondence should be addressed to the Editorial Board, Mathematical Medley, c/o Department of Mathematics, The National University of Singpaore, 10 Kent Ridge Crescent, Singapore 0511.

Problems

P18.2.1. *Proposed by Chan Heng Huat, National University of Singapore.*

Prove that the equation $2 \cdot 5^k = x^2 + 1$ has no integral solution for $k > 2$.

P18.2.2. *Proposed by Leong Chong Ming, Nanyang Junior College, Singapore.*

Estimate $\frac{1}{1}\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \ldots + (-1)^{n-1}\frac{1}{n}\binom{n}{n}$ for large n.

P18.2.3. *Proposed by Ng Weng Leong, Raffles Junior College, Singapore.*

The equation $a^b = b^a$ has the following integral solutions $a = 2, b = 4$ or $a = 4, b = 2$. Find other solutions, if any.

Solutions

P18.1.3. *The 1990 Asian Pacific Mathematical Olympiad question.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let S_k be the sum of products of $a_1, a_2, \ldots, a_n$ taken k at a time.

Show that

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \ldots a_n, \quad \text{for } k = 1, 2, \ldots, n-1.$$

Solution by Ng Weng Leong, Raffles Junior College, Singapore.

Let $S_k = \sum_{r=1}^{\binom{n}{k}} p_r$, where each p_r is a product of $a_1, \ldots, a_n$ with k factors. Then

$$S_{n-k} = \left(\prod_{r=1}^{n} a_r \right) \left(\sum_{r=1}^{\binom{n}{k}} \frac{1}{p_r} \right),$$

so that

$$\begin{aligned}
S_k S_{n-k} &= \left(\prod_{r=1}^{n} a_r \right) \left(\sum_{r=1}^{\binom{n}{k}} p_r \right) \left(\sum_{r=1}^{\binom{n}{k}} \frac{1}{p_r} \right) \\
&= \left(\prod_{r=1}^{n} a_r \right) \left[\binom{n}{k} + \sum_{\substack{i=1, j=1 \\ i \neq j}}^{\binom{n}{k}} \left(\frac{p_i}{p_j} + \frac{p_j}{p_i} \right) \right] \\
&\geq \left(\prod_{r=1}^{n} a_r \right) \left[\binom{n}{k} + \sum_{\substack{i=1, j=1 \\ i \neq j}}^{\binom{n}{k}} 2 \right] \\
&\geq \left(\prod_{r=1}^{n} a_r \right) \left[\binom{n}{k} + \binom{n}{k} \left(\binom{n}{k} - 1 \right) \right] \\
&= \binom{n}{k}^2 \left(\prod_{r=1}^{n} a_r \right).
\end{aligned}$$

P18.1.4. *The 1990 Asian Pacific Mathematical Olympiad question.*

Consider all triangles ABC which have a fixed base AB and whose altitude from C is a constant h. For which of these triangles is the product of its altitudes a maximum?

Solution by Ng Weng Leong, Raffles Junior College, Singapore.

Let h_a and h_b be the altitudes from A and B, respectively.

then

$$2 \cdot \text{area of } \triangle ABC = AB \cdot h = AC \cdot h_b = BC \cdot h_a = BC \cdot AC \cdot \sin C.$$

Therefore

$$(h \cdot h_b \cdot h_a)(AB \cdot AC \cdot BC) = (AB \cdot h)^3.$$

Consequently

$$(h \cdot h_b \cdot h_a)(AB \cdot AB \cdot h) = (AB \cdot h)^3 \sin C.$$

It implies

$$(h \cdot h_b \cdot h_a) = ABh^2 \sin C.$$

So the product $h \cdot h_a \cdot h_b$ attains its maximum when $\sin C$ reaches its maximum.

There are two cases :

(a) If $h \leq AB/2$, then there exists a triangle ABC which has a right angle at C where $\sin C$ reaches its maximum, namely 1 (Figure 1).

(b) If $h > AB/2$, the angle at C is acute and assumes its maximum when the triangle is isosceles (Figure 2).

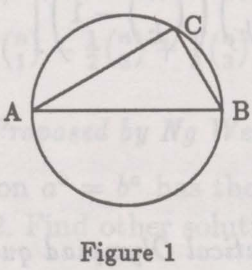

Figure 1

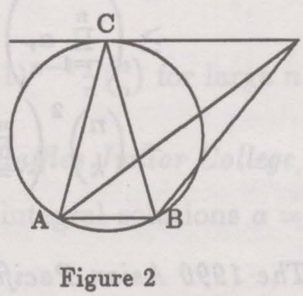

Figure 2