The Alexander-Conway polynomial of the Generalized Hopf link

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§1. Introduction

For each oriented link $L$, a Laurent polynomial $\Delta_L(t)$ with integral coefficients is uniquely determined by the following two axioms:

(I) $\Delta_U(t) = 1$, where $U$ is the unknot.

(II) For any 3 links $L_+, L_-$ and $L_0$ which are identical except within a small region where they have projections as below:

\[
\begin{align*}
\Delta_{L_+}(t) - \Delta_{L_-}(t) &= (\sqrt{t} - \frac{1}{\sqrt{t}})\Delta_{L_0}(t).
\end{align*}
\]

$\Delta_L$ is called the Alexander-Conway Polynomial of $L$.

In this paper we try to calculate the Alexander-Conway polynomial of the following link.
This is a multiple cable of the Hopf link. We denote this link by $L(m, n)$. Recently it has been shown in [1] that knot polynomials of cables of links play an important role in the calculation of some new invariants of 3-manifolds. We are able to calculate the Alexander - Conway polynomial of $L(2, n)$. The general case is still to be determined.

§2. The Generalized Hopf link

Let $Q(n)$ be the link shown on the right. Here the dotted string represents $n$ parallel strings all oriented in the same sense. In order to calculate $\Delta_{L(2,n)}(t)$, the following skein tree is derived.
For simplicity we denote $\Delta_{L(2,n)}(t)$ by $P_n$ and $\Delta_{Q(n)}$ by $Q_n$. Using axiom II and the fact that $\Delta_{\text{SplitLink}}(t) = 0$, we have from the above diagram the following relation.

\[
(1) \quad P_n = -2(\sqrt{t} - \frac{1}{\sqrt{t}})P_{n-1} + (\sqrt{t} - \frac{1}{\sqrt{t}})^2Q_{n-1}
\]
Theorem 1
(i) \( \Delta_{L(1,n)}(t) = (-1)^n(\sqrt{t} - \frac{1}{\sqrt{t}})^n \)
(ii) \( \Delta_{L(2,2)}(t) = -\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3(2 + t^{-1} + t) \)
(iii) \( \Delta_{L(2,3)}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})^4(2 + t^{-1} + t)(1 + t^{-1} + t) \)

Proof Let's only prove (iii). We first calculate \( \Delta_{Q(2)}(t) \). From figure A we obtain \( \Delta_{Q(2)}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})^2(2 + t^{-1} + t)(-1 + t^{-1} + t) \). Using (1) with \( n = 3 \) we have the required result.

Theorem 2
\[
\Delta_{L(2,n)}(t) = \frac{(-1)^{n+1}(\sqrt{t} - \frac{1}{\sqrt{t}})^n(1 - t^n)(1 + t)^{n-1}}{(1 - t)t^{n-1}}.
\]

Proof The link \( Q(n-1) \) is the closure of the braid \( \sigma = [\sigma_1 \sigma_2 ... \sigma_{n-1} \sigma_{n-1} \sigma_2 \sigma_1]^2 \), where \( \sigma_i \) is the ith elementary braid of the n-string braid group. Then \( \text{det}[B(\sigma) - I_n](1 + t + ... + t^{n-1})^{-1} \) gives the Alexander polynomial of \( Q(n-1) \), where \( B(\sigma) \) is the Burau matrix of \( \sigma \). From this, the Alexander polynomial of \( Q(n-1) \) can be calculated and is equal to \( (t - 1)^{n-1}[t^n + 1][t + 1]^{n-2} \). In the normal form which is the Alexander-Conway polynomial of \( Q(n-1) \), we have
\[
Q_{n-1} = (-1)^{n-1}(\sqrt{t} - \frac{1}{\sqrt{t}})^n(\sqrt{t}^n - (\frac{1}{\sqrt{t}})^n)[\sqrt{t} + \frac{1}{\sqrt{t}}]^{n-2}, \quad n > 0
\]
Substituting this into (1), we have
\[
P_n = -2(\sqrt{t} - \frac{1}{\sqrt{t}})P_{n-1} + (-1)^{n-1}(\sqrt{t} - \frac{1}{\sqrt{t}})^n(\sqrt{t}^n - (\frac{1}{\sqrt{t}})^n)[\sqrt{t} + \frac{1}{\sqrt{t}}]^{n-2}.
\]
Since the total number of strings involved in \( L(2, n) \) is \( n+2 \), we expect \( P_n \) to have a factor \( (-1)^{n+1}(\sqrt{t} - \frac{1}{\sqrt{t}})^{n+1} \). By letting \( P'_n = P_n[(-1)^{n+1}(\sqrt{t} - \frac{1}{\sqrt{t}})^{n+1}]^{-1} \), we get
\[
P'_n = 2P'_{n-1} + [(\sqrt{t})^n - (\frac{1}{\sqrt{t}})^n][\sqrt{t} + \frac{1}{\sqrt{t}}]^{n-2}.
\]
This can be simplified to \( P'_n = 2P'_{n-1} + t(t+1)^{n-2} + \frac{1}{t}(1 + \frac{1}{t})^{n-2} \). From this we obtain
\[
P'_n = 2^{n-1} + 2^{n-2}(t + \frac{1}{t}) + 2^{n-3}[t(t+1) + (1 + \frac{1}{t}) - \frac{1}{t}] + 2^{n-4}[t(t+1)^2 + (1 + \frac{1}{t})^2 - \frac{1}{t}]
\]
\[
+ ... + 2^{-k}[t(t+1)^{k-2} + (1 + \frac{1}{t})^{k-2} - \frac{1}{t}] + ... + [t(t+1)^{n-2} + (1 + \frac{1}{t})^{n-2} - \frac{1}{t}].
\]
Hence
\[
P'_n = 2^{n-1} + \frac{t[2^{n-1} - (t + 1)^{n-1}]}{(1 - t)} + \frac{[2^{n-1} - (1 + \frac{1}{t})^{n-1}]}{(t - 1)}.
\]
Therefore
\[ P_n = \frac{(-1)^{n+1}(\sqrt{i} - \frac{1}{\sqrt{i}})^{n+1}(1 - t^n)(1 + t)^{n-1}}{(1 - t)^{n-1}}. \]

References

[1] W.B.R. Lickorish, Invariants for 3-manifolds from the combinatoric of the Jones polynomial preprint