The Alexander-Conway polynomial of the Generalized Hopf link *

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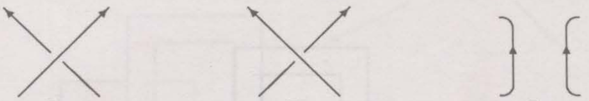
§1. Introduction

For each oriented link L, a Laurent polynomial $\Delta_L(t)$ with integral coefficients is uniquely determined by the following two axioms:

(I) $\Delta_U(t) = 1$, where U is the unknot.

(II) For any 3 links L_+, L_- and L_o which are identical except within a small region where they have projections as below :

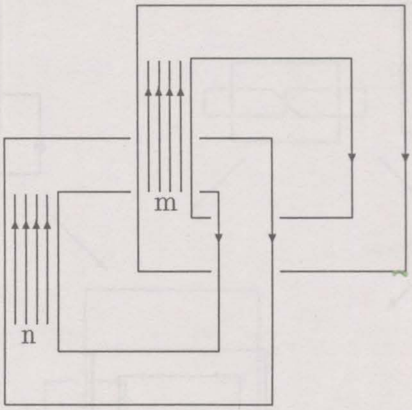


L_+ L_- L_o

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (\sqrt{t} - \tfrac{1}{\sqrt{t}})\Delta_{L_o}(t).$$

Δ_L is called the Alexander-Conway Polynomial of L.

In this paper we try to calculate the Alexander-Conway polynomial of the following link.



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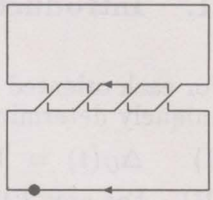
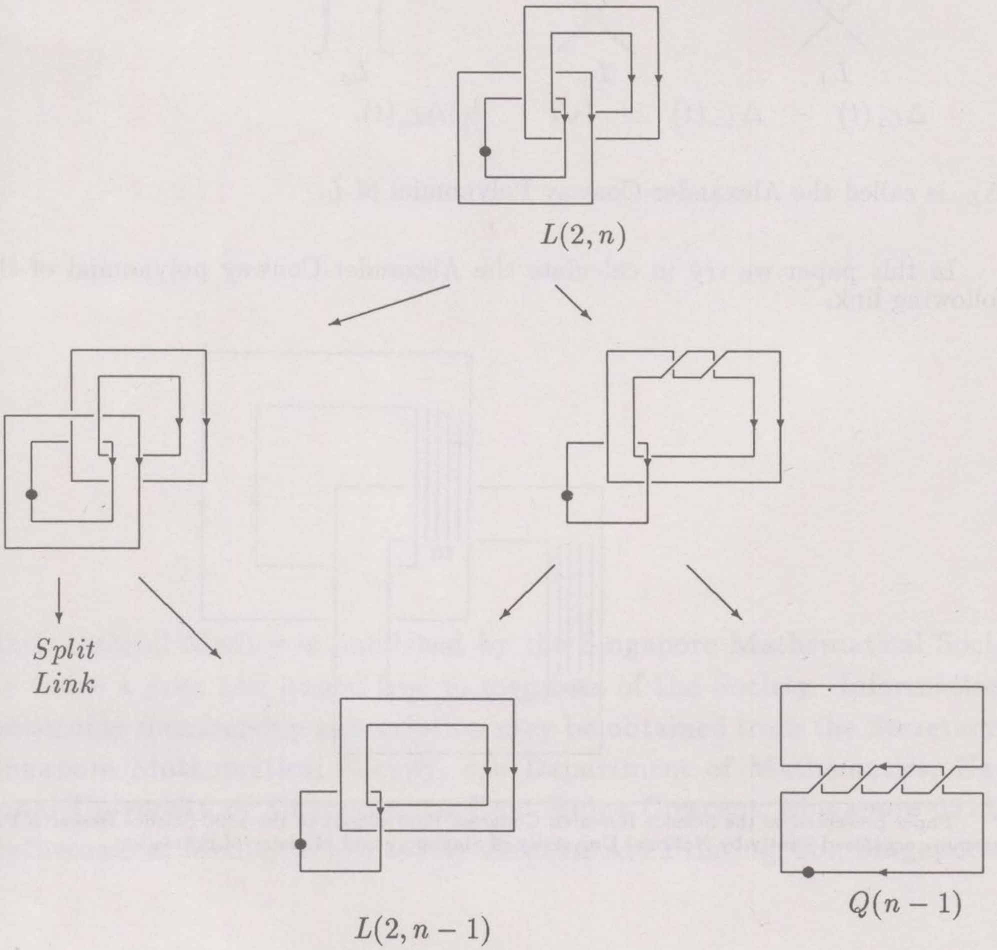
This is a multiple cable of the Hopf link. We denote this link by $L(m,n)$. Recently it has been shown in [1] that knot polynomials of cables of links play an important role in the calculation of some new invariants of 3-manifolds. We are able to calculate the Alexander - Conway polynomial of $L(2,n)$. The general case is still to be determined.

§2. The Generalized Hopf link

Let $Q(n)$ be the link shown on the right. Here the dotted string represents n parallel strings all oriented in the same sense. In order to calculate $\Delta_{L(2,n)}(t)$, the following skein tree is derived.



$Q(n)$



For simplicity we denote $\Delta_{L(2,n)}(t)$ by P_n and $\Delta_{Q(n)}$ by Q_n. Using axiom II and the fact that $\Delta_{SplitLink}(t) = 0$, we have from the above diagram the following relation.

$$(1) \quad P_n = -2(\sqrt{t} - \tfrac{1}{\sqrt{t}})P_{n-1} + (\sqrt{t} - \tfrac{1}{\sqrt{t}})^2 Q_{n-1}$$

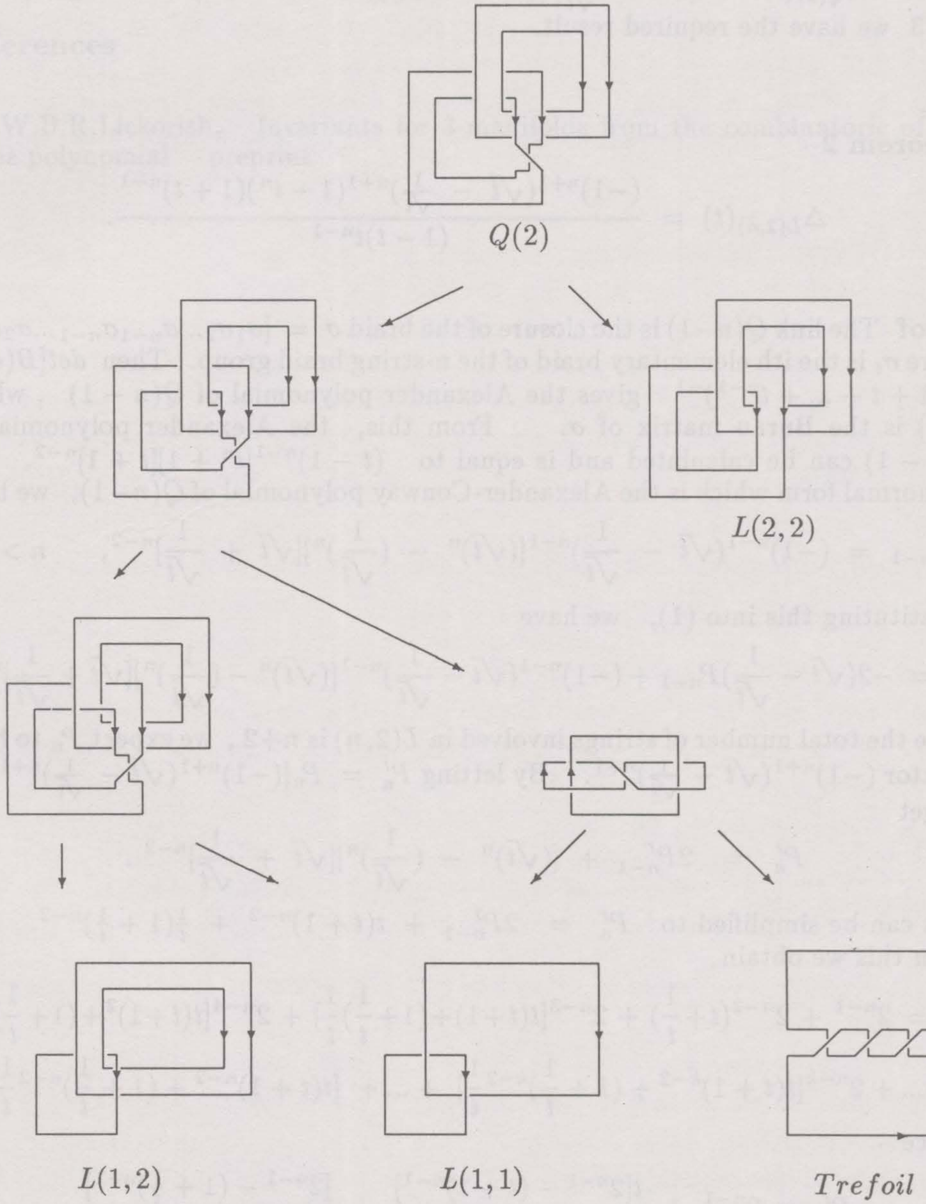


Figure A

Theorem 1
(i) $\Delta_{L(1,n)}(t) = (-1)^n(\sqrt{t} - \frac{1}{\sqrt{t}})^n$
(ii) $\Delta_{L(2,2)}(t) = -(\sqrt{t} - \frac{1}{\sqrt{t}})^3(2+t^{-1}+t)$
(iii) $\Delta_{L(2,3)}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})^4(2+t^{-1}+t)(1+t^{-1}+t)$

Proof Let's only prove (iii). We first calculate $\Delta_{Q(2)}(t)$. From figure A we obtain $\Delta_{Q(2)}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})^2(2+t^{-1}+t)(-1+t^{-1}+t)$. Using (1) with $n=3$ we have the required result.

Theorem 2

$$\Delta_{L(2,n)}(t) = \frac{(-1)^{n+1}(\sqrt{t} - \frac{1}{\sqrt{t}})^{n+1}(1-t^n)(1+t)^{n-1}}{(1-t)t^{n-1}}.$$

Proof The link $Q(n-1)$ is the closure of the braid $\sigma = [\sigma_1\sigma_2...\sigma_{n-1}\sigma_{n-1}...\sigma_2\sigma_1]^2$, where σ_i is the ith elementary braid of the n-string braid group. Then $det[B(\sigma)-I_n](1+t+...+t^{n-1})^{-1}$ gives the Alexander polynomial of $Q(n-1)$, where $B(\sigma)$ is the Burau matrix of σ. From this, the Alexander polynomial of $Q(n-1)$ can be calculated and is equal to $(t-1)^{n-1}[t^n+1][t+1]^{n-2}$. In the normal form which is the Alexander-Conway polynomial of $Q(n-1)$, we have

$$Q_{n-1} = (-1)^{n-1}(\sqrt{t} - \frac{1}{\sqrt{t}})^{n-1}[(\sqrt{t})^n - (\frac{1}{\sqrt{t}})^n][\sqrt{t} + \frac{1}{\sqrt{t}}]^{n-2}, \qquad n > 0$$

Substituting this into (1), we have

$$P_n = -2(\sqrt{t} - \frac{1}{\sqrt{t}})P_{n-1} + (-1)^{n-1}(\sqrt{t} - \frac{1}{\sqrt{t}})^{n-1}[(\sqrt{t})^n - (\frac{1}{\sqrt{t}})^n][\sqrt{t} + \frac{1}{\sqrt{t}}]^{n-2}.$$

Since the total number of strings involved in $L(2,n)$ is $n+2$, we expect P_n to have a factor $(-1)^{n+1}(\sqrt{t} - \frac{1}{\sqrt{t}})^{n+1}$. By letting $P'_n = P_n[(-1)^{n+1}(\sqrt{t} - \frac{1}{\sqrt{t}})^{n+1}]^{-1}$, we get

$$P'_n = 2P'_{n-1} + [(\sqrt{t})^n - (\frac{1}{\sqrt{t}})^n][\sqrt{t} + \frac{1}{\sqrt{t}}]^{n-2}.$$

This can be simplified to $P'_n = 2P'_{n-1} + t(t+1)^{n-2} + \frac{1}{t}(1+\frac{1}{t})^{n-2}$.
From this we obtain

$$P'_n = 2^{n-1} + 2^{n-2}(t+\frac{1}{t}) + 2^{n-3}[t(t+1)+(1+\frac{1}{t})\frac{1}{t}] + 2^{n-4}[t(t+1)^2+(1+\frac{1}{t})^2\frac{1}{t}]$$
$$+ ... + 2^{n-k}[t(t+1)^{k-2} + (1+\frac{1}{t})^{k-2}\frac{1}{t}] + ... + [t(t+1)^{n-2} + (1+\frac{1}{t})^{n-2}\frac{1}{t}].$$

Hence

$$P'_n = 2^{n-1} + \frac{t[2^{n-1} - (t+1)^{n-1}]}{(1-t)} + \frac{[2^{n-1} - (1+\frac{1}{t})^{n-1}]}{(t-1)}.$$

$$= \frac{[(1-t^n)(1+t)^{n-1}]}{[(1-t)t^{n-1}]}.$$

Therefore

$$P_n = \frac{(-1)^{n+1}(\sqrt{t} - \frac{1}{\sqrt{t}})^{n+1}(1-t^n)(1+t)^{n-1}}{(1-t)t^{n-1}}.$$

References

[1] W.B.R.Lickorish, Invariants for 3-manifolds from the combinatoric of the Jones polynomial preprint