Wavelets

Toh Kim Chuan
National University of Singapore

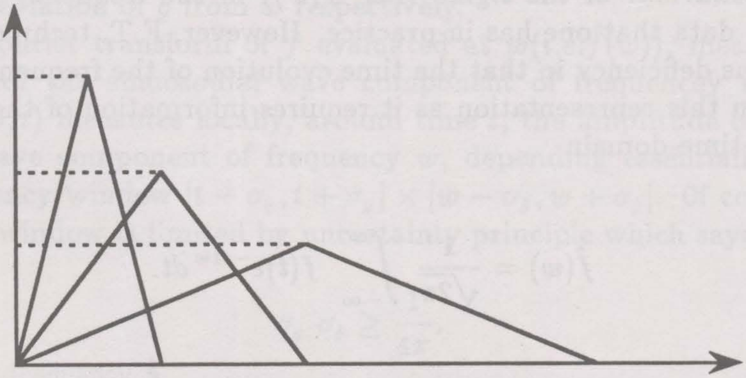
A wavelet basis is a sequence of functions that is generated from a single function ψ, called mother wavelet, by taking combinations of translates and dilates of ψ. More specifically, it has the form

$$\{\sqrt{a^m}\ \psi(a^m \chi - bn)\}_{m,n\in\mathbb{Z}},$$

where $a, b > 0$ are fixed constants. As an example, for $a = 2, b = 1$ and ψ given by

$$\psi(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 2 - x & 1 \leq x \leq 2 \\ 0 & x > 2, \end{cases}$$

the sequence $\{\sqrt{2^m}\ \psi(2^m - n)\}_{m,n\in\mathbb{Z}}$ is a wavelet basis.



Like Fourier analysis, wavelet basis allows decomposition of functions into coefficients. We will discuss later the advantages of wavelet decomposition over the conventional Fourier decomposition.

Wavelets were introduced in France in the early 1980s by Jean Morlet, a geophysicist and Alexander Grossman, a mathematical physicist, to analyse seismic signal. The mathematical theory of wavelets took off in 1985 when Yves Meyer, also in France, constructed the first orthogonal system of smooth wavelets such that their fourier transform have compact support. In 1986, Meyer and Stephane Mallat developed the theory of multiresolution analysis that provides a natural framework for the theory of wavelet approximations and construction of orthonormal wavelet basis. I. Daubechies, in 1987, constructed orthogonal systems of compactly supported wavelets (the size of the support grows linearly with the degree of smoothness). Since then, this area has flourished.

The potential applications of the wavelet theory in mathematics, engineering and physics explains why it has attracted so much attention. The theory has shown great promises in the areas of pure and applied mathematics like approximation theory, harmonic analysis, operator theory, numerical partial differential equations, etc. Much of the interest in the engineering side is in the applications to signal processing (ranging from image processing, acoustic signal, seismic signal to synthetic music) where wavelets seem to hold great promise for detection of edges and singularities, providing efficient decomposition and reconstruction algorithm for signals, and data compression. Not to forget, physicists are already using wavelets in quantum mechanics and quantum field theory.

Why wavelets come into the scene at all, a natural question one would ask. It is well known that the representation of a signal $f(t)$ (acoustic, electrical, etc) by means of its spectrum (or Fourier transform) is essential to solve many problems in engineering and mathematics. In fact, the spectral behaviour of the signal (i.e. $\hat{f}(w)$) in the frequency domain is the actual data that one has in practice. However, F.T. techniques has a very serious deficiency in that the time evolution of the frequencies is not reflected in this representation as it requires information of the signal in the entire time-domain

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-itw}\, dt.$$

One can see that, if $f(t)$ is perturbed by an impulse at time $t = t_0$, $\hat{f}(w)$ would change correspondingly but it does not tell us when is $f(t)$ being perturbed. It is important to know this if one is to edit out this unwanted perturbation in $f(t)$ such as an attack of a musical note.

Noticing this deficiency, D Gabor, in his 1946 paper, introduced a time-frequency localization method (called short-time Fourier transform, STFT) by introducing a window function g to "window" the Fourier integral

$$Gf(w,t) = \int_{-\infty}^{\infty} f(t')e^{-iwt'} g(t'-t)dt'$$

$$= \int_{-\infty}^{\infty} \hat{f}(w')e^{-i(w-w')t} \hat{g}(w'-w)dw'.$$

From these two integrals, we see that $G(w,t)$ depends essentially on $f(t')$ for $t' \in [t-\sigma_g, t+\sigma_g]$ and $\hat{f}(w')$ for $w' \in [w-\sigma_{\hat{g}}, w+\sigma_{\hat{g}}]$ in the time and frequency domain respectively. We have chosen g to be real-valued function such that

$$\int_{-\infty}^{\infty} |g(t)|^2\, dt = 1, \quad \int_{-\infty}^{\infty} t|g(t)|^2\, dt = 0$$

and σ_g, $\sigma_{\hat{g}}$ are respectively the standard deviation of g and $\hat{g}$,

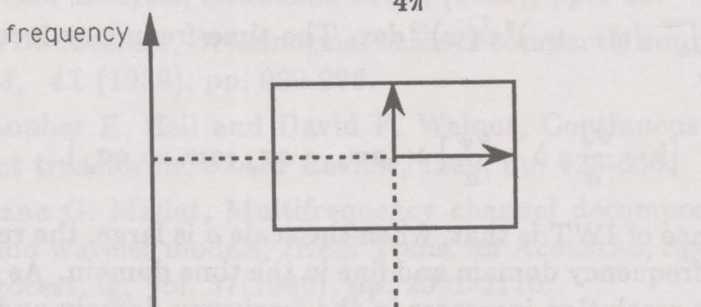
$$\sigma_g^2 = \int_{-\infty}^{\infty} t^2 |g(t)|^2\, dt$$

$$\sigma_{\hat{g}}^2 = \int_{-\infty}^{\infty} w^2 |\hat{g}(w)|^2\, dw.$$

We assumed that the dependent of $Gf(w,t)$ on $f(t')$ and $\hat{f}(w')$ is significant only for t' within a standard deviation of g from t and w' within a standard deviation of $\hat{g}$ from w respectively.

The Fourier transform of f evaluated at $w(i.e.\hat{f}(w))$, measures the amplitude of the sinuosodial wave component of frequency w. Likewise, $Gf(w,t)$ measures locally, around time t, the amplitude of the sinuosoidal wave component of frequency w, depending essentially on the time-frequency window $[t-\sigma_g, t+\sigma_g] \times [w-\sigma_{\hat{g}}, w+\sigma_{\hat{g}}]$. Of course, the size of the window is limited by uncertainty principle which says that

$$\sigma_g\, \sigma_{\hat{g}} \geq \frac{1}{4\pi}.$$



Suppose $f(t)$ is perturbed by an impulse at $t = t_0$, then this would be reflected in $Gf(w,t)$ for $t \in [t_0 - \sigma_g, t_0 + \sigma_g]$. Thus, the information provided by this decomposition is therefore unlocalized within intervals of size σ_g. Similar conclusion holds in the frequency domain. If a signal has a discontinuity such as an edge, it is difficult to locate it with a precision better than σ_g. In general, for edge detection, the time-window must be very narrow at a high-frequency band (for the location of an edge) for accuracy, and very wide at a low-frequency band for efficiency. Because the time and frequency resolution of STFT is constant (i.e. the size of the time and frequency windows are independent of t and w), it is impossible to define an optimal resolution for analysing signal that has important features of very different sizes. This is particularly the case with images, for example, in the image of a house, the pattern we want to analyse might range from the overall structure of the house (low frequency band) to the details on one of the curtains (high frequency band)

To overcome the inflexibility of fixed time-frequency resolution of STFT, A Grossman and J. Marlet introduced integral wavelet transform (IWT) in 1984.

$$\begin{aligned} Wf(a,b) &= \sqrt{a} \int_{-\infty}^{\infty} f(t)\psi(a(t-b))dt \\ &= \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \hat{f}(w)e^{ibw}\ \hat{\psi}(\frac{w}{a})dw, \end{aligned}$$

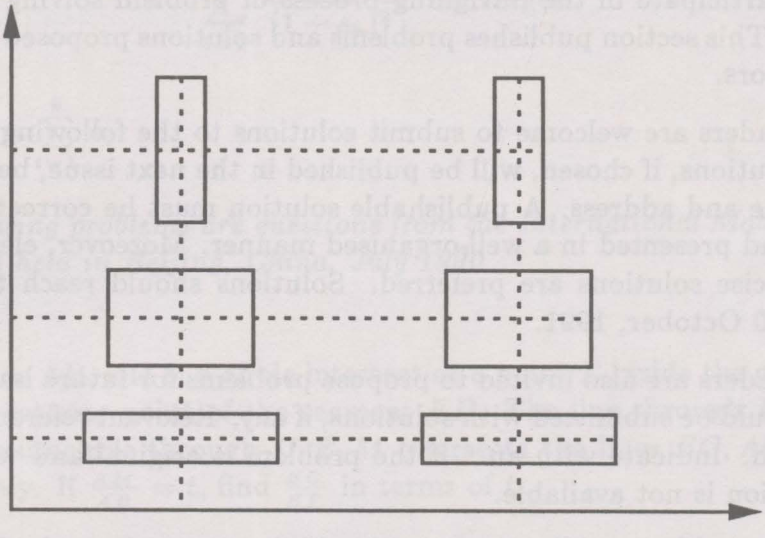
where ψ is a chosen fixed real-valued function such that $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$, $\int_{-\infty}^{\infty} t|\psi(t)|^2 dt = 0$ and $\int_{-\infty}^{\infty} \psi(t)dt = 0$. The first integral shows that $Wf(a,b)$ depends on $f(t)$ essentially for $t \in [b - \frac{\sigma_\psi}{a},\ b + \frac{\sigma_\psi}{a}]$, where $\sigma_\psi^2 = \int_{-\infty}^{\infty} t^2|\psi(t)|^2 dt$. Let $w_0 = \int_0^{\infty} w|\hat{\psi}(w)|^2 dw$. In practice, $|\hat{f}(w)| = 0$ for $w < 0$, the second integral shows that $Wf(a,b)$ depends on $\hat{f}(w)$ essentially for

$$w \in [aw_0 - a\sigma_{\hat{\psi}},\ aw_0 + a\sigma_{\hat{\psi}}],$$

where $\sigma_{\hat{\psi}}^2 = \int_0^{\infty} (w-w_0)^2 \hat{\psi}(w)|^2 dw$. The time-frequency localization is thus given by

$$[b - \frac{\sigma_\psi}{a},\ b + \frac{\sigma_\psi}{a}] \times [aw_0 - a\sigma_{\hat{\psi}}, aw_0 + a\sigma_{\hat{\psi}}].$$

The significance of IWT is that, when the scale a is large, the resolution is coarse in the frequency domain and fine in the time domain. As the scale a decreases, the resolution increases in the frequency domain and decreases in the time domain. This variation of resolution enables the IWT to zoom into the details of a function in a way STFT cannot, (identify a with a constant multiple of the frequency) giving sharper time resolution at higher frequencies and efficiency at low frequencies.



Time-frequency localization windows of IWT

As wavelet decomposition separates and localize the spectral information in different frequency bands, hence filtering, detection, data reduction, enhancement can be easily implemented before applying the wavelet reconstruction algorithm. And it is this capability of wavelets that has made it a star in the engineering, physical and mathematical communities.

References

[1] Charles K. Chui, An overview of wavelets, approximation theory and functional analysis, *Academic Press*, (1990), pp.1-26.

[2] Ingrid Daubechies, Orthonormal bases of compactly supported wavelets, *CPAM*, **41** (1988), pp. 909-996.

[3] Christopher E. Heil and David F. Walnut, Continuous and discrete wavelet transforms, *SIAM Review*, 1989, pp. 628-666.

[4] Stephane G. Mallat, Multifrequency channel decompositions of images and wavelet models, *IEEE Trans. on Acoustics*, Speech and Signal Processing, Vol. 37(1989), pp. 2091-2110.