Inequalities associated with the Principle of Inclusion and Exclusion

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The 30th International Mathematical Olympiad (IMO) was held in the Federal Republic of Germany in July, 1989. The 6th problem in this competition is as follows:

> A permutation $x_1 x_2 ... x_{2n}$ of the set $\{1, 2, ..., 2n\}$, where n is a natural number, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, ..., n-1\}$. Show that for each n there are more permutations with property P than without.

The solution of the above problem given by the proposer involves some complicated recurrence relations, which make the problem looks hard. Soon after it was selected as an IMO problem, and before the competition was held, some jury members found another way of solving the problem by applying the Principle of Inclusion and Exclusion (PIE). We shall now see how this can be done.

Since the problem is trivial when $n = 1$, we assume that $n \geq 2$. Let $S = \{1, 2, ..., 2n\}$. Evidently, a permutation of S satisfies property P if and only if the pair of numbers $\{r, n+r\}$ are adjacent in the permutation for some $r = 1, 2, ..., n$. Now, for each $r = 1, 2, ..., n$, let A_r be the set of permutations of S in which r and $n+r$ are adjacent. If we denote by $f(n)$ (respectively $g(n)$) the number of permutations with property P (respectively without P), then

$$f(n) = |\bigcup_{r=1}^{n} A_r| \tag{1}$$

and our aim is to show that

$$f(n) > g(n) \tag{2}$$

for each $n \geq 2$.

The PIE says that

$$|\bigcup_{r=1}^{n} A_r| = \sum_{r=1}^{n} |A_r| - \sum_{1\le r<s\le n} |A_r \cap A_s| + \sum_{1\le r<s<t\le n} |A_r \cap A_s \cap A_t|$$
$$- \ldots + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|. \tag{3}$$

However, it is difficult to compute the exact value of (1) using (3), and (3) may not be useful either for proving (2). Actually, to prove (2), what we need is a good lower bound for (1) which, at the same time, can be evaluated without too much difficulty. We shall see that the lower bound for (1) given below serves the purpose :

$$|\bigcup_{r=1}^{n} A_r| \ge \sum_{r=1}^{n} |A_r| - \sum_{1\le r<s\le n} |A_r \cap A_s|. \tag{4}$$

Indeed, as

$$|A_r| = 2\cdot(2n-1)!$$

and for $r < s$,

$$|A_r \cap A_s| = 2^2\cdot(2n-2)!,$$

we have, by (1) and (4),

$$\begin{aligned} f(n) = |\bigcup_{r=1}^{n} A_r| &\ge n\cdot 2\cdot(2n-1)! - \binom{n}{2}\cdot 2^2\cdot(2n-2)! \\ &= 2n^2\cdot(2n-2)!; \\ \text{i.e.,}\quad f(n) &\ge 2n^2\cdot(2n-2)!. \end{aligned} \tag{5}$$

Since $f(n)+g(n)$ counts the number of permutations of S, it follows that

$$f(n)+g(n) = (2n)!. \tag{6}$$

Accordingly, we have

$$\begin{aligned} &f(n)-g(n) \\ &= f(n) - \{(2n)! - f(n)\} && \text{(by (6))} \\ &= 2f(n) - (2n)! \\ &\ge 4n^2\cdot(2n-2)! - (2n)! && \text{(by (5))} \\ &= 2n\cdot(2n-2)! > 0 \end{aligned}$$

for $n \geq 2$. Thus equality (2) holds, as required.

While inequality (4) is essential in the above proof, the inequality itself is not familiar to many of us. Recently, by using a standard argument of determining the count contributed by an element to a quantitative expression, Jiang [2] proved that inequality (4) holds and further pointed out that the same argument could be used to prove a more general inequality related to (3). To see this and to simplify the statement of this general result, we introduce the following notation.

In the remainder of this note, let S be the universal set, and let $A_1, A_2, \ldots, A_n$ be any n subsets of S. Define

$$w(0) = |S|,$$

and for each integer r with $1 \leq r \leq n$,

$$w(r) = \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} |A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r}|.$$

Thus (3) can be re-stated simply as

$$|\bigcup_{r=1}^{n} A_r| = \sum_{r=1}^{n} (-1)^{r+1}\, w(r). \tag{3'}$$

The general inequalities observed by Jiang are as follows:
For each $k = 1, 2, \ldots, n$,

$$|\bigcup_{r=1}^{n} A_r| \begin{Bmatrix} \geq \\ \leq \end{Bmatrix} \sum_{r=1}^{k} (-1)^{r+1}\, w(r) \text{ if } k \text{ is } \begin{cases} \text{even} \\ \text{odd.} \end{cases} \tag{7}$$

Thus (4) is a special case of (7) when $k = 2$.

As a matter of fact, inequalities (7) and their proofs by mathematical induction can be found in [5] (see p.14 and p.119). Besides, it had been pointed out by Stanley in his book [4] (see p.91) that inequalities (7) can be deduced from an inequality due to Bonferroni [1] which states that for each $j = 0, 1, \ldots, n$,

$$\sum_{r=j}^{n} (-1)^{r-j}\, w(r) \geq 0. \tag{8}$$

We would like to add here that (8) can also be deduced from (7). First of all, note that (8) holds trivially when $j = 0$ or 1 as

$$\sum_{r=0}^{n} (-1)^r w(r) = w(0) - \sum_{r=1}^{n} (-1)^{r+1}\, w(r)$$

$$= |S| - |\bigcup_{r=1}^{n} A_r| = |S \setminus \bigcup_{r=1}^{n} A_r| \geq 0$$

and

$$\sum_{r=1}^{n} (-1)^{r-1}\ w(r) = \sum_{r=1}^{n} (-1)^{r+1} w(r) = |\bigcup_{r=1}^{n} A_r| \geq 0.$$

The equivalence of (7) and (8) now follows readily from the following observation:

$$\begin{aligned}
|\bigcup_{r=1}^{n} A_r| &= \sum_{r=1}^{n} (-1)^{r+1}\ w(r) \\
&= \sum_{r=1}^{k} (-1)^{r+1} w(r) + \sum_{r=k+1}^{n} (-1)^{r+1} w(r) \\
&= \sum_{r=1}^{k} (-1)^{r+1} w(r) + (-1)^{k+2} \sum_{r=k+1}^{n} (-1)^{r-(k+1)} w(r),
\end{aligned}$$

where $1 \leq k \leq n$.

Our main aim in this note is to point out that inequalities (7) and (8) can be further extended. For this purpose, we first state a principle which extends PIE.

For each integer m with $0 \leq m \leq n$, let $E(m)$ denote the number of elements in S which belong to *exactly* m of the n sets $A_1, A_2, \ldots, A_n$. Then the Generalized Principle of Inclusion and Exclusion (GPIE) states that (see, for instance, Liu [3])

$$E(m) = \sum_{r=m}^{n} (-1)^{r-m} \binom{r}{m} w(r). \tag{9}$$

By letting $m = 0$ in (9), we have

$$\begin{aligned}
|\bigcup_{r=1}^{n} A_r| &= |S \setminus \bigcap_{r=1}^{n} \bar{A}_r| = |S| - |\bigcap_{r=1}^{n} \bar{A}_r| \\
&= w(0) - E(0) \\
&= w(0) - \sum_{r=0}^{n} (-1)^r w(r) \\
&= \sum_{r=1}^{n} (-1)^{r+1} w(r),
\end{aligned}$$

where $\bar{A}_r$ denotes the complement of A_r in S. Thus (3′) is a special case of (9).

Before proceeding any further, we give here an example to illustrate identity (9).

Example 1. Let $S = \{1, 2, ..., 14\}$, and let A_1, A_2, A_3, A_4 be the subsets of S defined in Table 1.

S	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A_1	√			√				√	√			√		
A_2	√	√	√	√		√		√		√	√		√	
A_3	√		√					√	√		√			√
A_4	√		√					√		√	√			√

Table 1

Thus $A_1 = \{1,4,8,9,12\}$, $A_2 = \{1,2,3,4,6,8,10,11,13\}$ and so on. By definition, we have

$$\begin{aligned}
w(0) &= |S| = 14,\\
w(1) &= |A_1| + |A_2| + |A_3| + |A_4|\\
&= 5 + 9 + 6 + 6 = 26,\\
w(2) &= |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3|\\
&\quad + |A_2 \cap A_4| + |A_3 \cap A_4|\\
&= 3 + 3 + 2 + 4 + 5 + 5 = 22,\\
w(3) &= |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4|\\
&\quad + |A_2 \cap A_3 \cap A_4|\\
&= 2 + 2 + 2 + 4 = 10, \text{ and}\\
w(4) &= |A_1 \cap A_2 \cap A_3 \cap A_4| = 2.
\end{aligned}$$

Also,

$$\begin{aligned}
E(0) &= 2 \quad \text{(elements } 5, 7),\\
E(1) &= 4 \quad (2, 6, 12, 13),\\
E(2) &= 4 \quad (4, 9, 10, 14),\\
E(3) &= 2 \quad (3, 11), \text{ and}\\
E(4) &= 2 \quad (1, 8).
\end{aligned}$$

Suppose, for instance, $m = 2$. Then $E(2) = 4$ and

$$\begin{aligned}
&\sum_{r=m}^{n} (-1)^{r-m} \binom{r}{m} w(r) \\
&= \sum_{r=2}^{4} (-1)^{r-2} \binom{r}{2} w(r) \\
&= \binom{2}{2} w(2) - \binom{3}{2} w(3) + \binom{4}{2} w(4) \\
&= 22 - 3 \cdot 10 + 6 \cdot 2 \\
&= 4;
\end{aligned}$$

i.e., the two quantities on both sides of (9) agree.

We leave it to the readers to verify that equality (9) holds for $m = 0, 1, 3$.

For any integer k with $m \leq k \leq n$, let

$$A(m,k) = \sum_{r=m}^{k} (-1)^{r-m} \binom{r}{m} w(r).$$

Then GPIE simply says that

$$E(m) = A(m,n).$$

A related question arises. What is the relation between $E(m)$ and $A(m,k)$ when $k \leq n-1$?

Two integers a and b are said to have the *same parity* if and only if $a \equiv b \pmod 2$ (i.e., a and b are both even or both odd). A complete answer to the above question is now given below.

Theorem 1. Let m, k and n be integers with $0 \leq m \leq k \leq n-1$.

(i) If m and k have the same parity, then

$$E(m) \leq A(m,k).$$

(ii) If m and k have different parities, then

$$E(m) \geq A(m,k).$$

Furthermore, in each case, strict inequality holds if and only if $w(t) > 0$ for some t with $k < t \leq n$.

Example 2. As shown in Table 2, the inequalities stated in Theorem 1 are verified using the data given in Example 1. Note that the values of $A(m,k)$, where m and k have the same parity, are underlined.

m	$A(m,0)$	$A(m,1)$	$A(m,2)$	$A(m,3)$	$A(m,4)=E(m)$
0	$\underline{14}$	-12	$\underline{10}$	0	2
1		$\underline{26}$	-18	$\underline{12}$	4
2			$\underline{22}$	-8	4
3				$\underline{10}$	2

Table 2

To prove Theorem 1, we need some identities involving binomial coefficients given by the following lemma. Note that in this lemma, identity (i) is used to prove identity (iii).

Lemma 2. Let p, q and r be nonnegative integers with $p \leq q \leq r$. Then

(i) $\binom{q+1}{p+1} = \binom{q}{p} + \binom{q}{p+1}$ where $p < q$;

(ii) $\binom{r}{q}\binom{q}{p} = \binom{r}{p}\binom{r-p}{q-p}$;

(iii) $\sum_{j=0}^{p} (-1)^j \binom{q}{j} = \begin{cases} 0 & \text{if } p = q \\ (-1)^p \binom{q-1}{p} & \text{if } p < q. \end{cases}$

Proof. The two simple identities in (i) and (ii) follow easily from the formula:

$$\binom{q}{p} = \frac{q!}{p!(q-p)!}.$$

By letting $x = -1$ in the following binomial expansion:

$$(1+x)^p = \sum_{j=0}^{p} \binom{p}{j} x^j,$$

we obtain

$$\sum_{j=0}^{p} (-1)^j \binom{p}{j} = 0,$$

which is the first part of (iii).

To prove the second part of (iii), assume $p < q$. Observe that

$$\sum_{j=0}^{p} (-1)^j \binom{q}{j}$$

$$= 1 + \sum_{j=1}^{p} (-1)^j \left\{ \binom{q-1}{j-1} + \binom{q-1}{j} \right\} \quad \text{(by (i))}$$

$$= 1 - \left\{ \binom{q-1}{0} + \binom{q-1}{1} \right\} + \left\{ \binom{q-1}{1} + \binom{q-1}{2} \right\}$$

$$- \dots + (-1)^p \left\{ \binom{q-1}{p-1} + \binom{q-1}{p} \right\}$$

$$= 1 - \binom{q-1}{0} + (-1)^p \binom{q-1}{p}$$

$$= (-1)^p \binom{q-1}{p}. \quad \square$$

We are now in a position to prove Theorem 1.

Let x be an element of S. Assume that x belongs to *exactly* t of the n sets $A_1, A_2, \dots, A_n$, where $0 \le t \le n$. We shall compare the counts contributed by x to both $E(m)$ and $A(m, k)$ in each of the following cases which cover all the possibilities:

(a) $t < m$
(b) $t = m$
(c) $m < t \le k$
(d) $k < t$.

First of all, we note that x contributes to $w(r)$ a count of $\binom{t}{r}$ if $t \ge r$; and a count of 0 if $t < r$. With this in mind, we then have the following comparison.

	The count contributed by x to	
Case	$E(m)$	$A(m,k)$
(a)	0	0
(b)	1	$(-1)^{m-m}\binom{m}{m}\cdot 1 = 1$
(c)	0	$\sum_{r=m}^{t}(-1)^{r-m}\binom{r}{m}\binom{t}{r}$ $= \sum_{r=m}^{t}(-1)^{r-m}\binom{t}{m}\binom{t-m}{r-m}$ (Lemma 2(ii)) $= \binom{t}{m}\sum_{j=0}^{t-m}(-1)^{j}\binom{t-m}{j}$ ($j = r-m$) $= 0$ (Lemma 2(iii))
(d)	0	$\sum_{r=m}^{k}(-1)^{r-m}\binom{r}{m}\binom{t}{r}$ $= \binom{t}{m}\sum_{j=0}^{k-m}(-1)^{j}\binom{t-m}{j}$ (Lemma 2(ii)) $= (-1)^{k-m}\binom{t-m-1}{k-m}\binom{t}{m}$ (Lemma 2(iii))

Observe that the counts contributed by x to $E(m)$ and $A(m,k)$ are the same in all cases except case (d); and in this case, the count contributed by x to $A(m,k)$ is positive if and only if k and m have the same parity. The inequalities in (i) and (ii) thus follow. Finally, strict inequality holds in each case if and only if there is an element in S which satisfies the condition in case (d); i.e., $w(t) > 0$ for some t with $k < t \leq n$. □

Finally, we note that the result in (7) is a special case of Theorem 1. Indeed, when $m = 0$, we have

$$E(0) = w(0) - \left|\bigcup_{r=1}^{n} A_r\right|$$

and

$$A(0,k) = \sum_{r=0}^{k}(-1)^r w(r).$$

By Theorem 1,

$$E(0)\begin{cases} \leq & A(0,k) \text{ if } k \text{ is even} \\ \geq & A(0,k) \text{ if } k \text{ is odd;} \end{cases}$$

i.e.,

$$|\bigcup_{r=1}^{n} A_r| \left\{ \begin{matrix} \geq \\ \leq \end{matrix} \right\} \sum_{r=1}^{k} (-1)^{r+1} w(r) \quad \text{if } k \text{ is } \left\{ \begin{matrix} \text{even} \\ \text{odd,} \end{matrix} \right.$$

which is the result in (7).

References

[1] Bonferroni, *Public. Ist. Sup. Sc. Ec. Comm. Firenze* 8 (1936), 1-62.

[2] M-S Jiang, *Some probability problems arising from an IMO problem (in Chinese), Mathematics Olympiad 6-9 (1991)*, 236-241.

[3] C. L. Liu, *Introduction to Combinatorial Mathematics*, New York, McGraw-Hill, 1968.

[4] R. P. Stanley, *Enumerative Combinatorics (Volume 1)*, Wadsworth & Brooks, 1986.

[5] I. Tomescu, *Problems in Combinatorics and Graph Theory*, John Wiley & Sons, 1985.