Introduction

As a branch of mathematics, combinatorics is becoming increasingly important and lively. Combinatorial problems and their applications can be found not only in various branches of mathematics, but also in other disciplines such as sciences, operational research, statistics, management science, etc. Combinatorial techniques have also become powerful and essential tools for computer scientists. One of the general problems that combinatorics deals with is counting. Problems that have to do with counting occur everywhere, not only in advanced researches, but also in our everyday life. It would thus be useful for us to be familiar with some basic knowledge about counting. In this series of articles, we shall introduce a wide range of elementary counting problems. Some basic principles and efficient techniques used to solve the problems will also be discussed.

COUNTING

- Its Principles and Techniques (1) -

by K M Koh and B P Tan

1. The Addition Principle

We first begin with two basic principles: the Addition Principle in this section and the Multiplication Principle in the next section. We shall illustrate by examples how these principles are applied in the process of counting.

Consider a 4-element set $A = \{a, b, c, d\}$. In how many ways can we form a 2-element subset of A? This can be answered easily by simply listing all the 2-element subsets:

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}.$$

The answer is '6'.

Let us try a slightly more complicated problem.

Example 1.1

A group of students consists of 3 boys and 4 girls. How many ways are there to form a pair of students of the same sex from the group?

Naturally, we divided our consideration into two cases: both of the students in a pair are boys, or both are girls. Assume that the 3 boys are b_1, b_2 and b_3. Then there are 3 ways to form such a pair; namely,

$$\{b_1, b_2\}, \{b_1, b_3\}, \{b_2, b_3\}.$$

Suppose the 4 girls are g_1, g_2, g_3 and g_4. Then there are 6 ways to form such a pair; namely,

$$\{g_1, g_2\}, \{g_1, g_3\}, \{g_1, g_4\}, \{g_2, g_3\}, \{g_2, g_4\}, \{g_3, g_4\}.$$

Thus the desired number of ways is equal to $3 + 6 = 9$.

In dealing with the counting problems, quite often, we have to divide our consideration into cases which are *disjoint* (like boys or girls in Example 1.1) and *exhaustic* (besides boys or girls, no other cases remain). Then the total number of ways would be the sum of the numbers of ways in the cases. More precisely, we have:

The Addition Principle
Suppose that there are n_1 ways for the event E_1 to occur and n_2 ways for the event E_2 to occur. If all these ways are pairwise distinct, then the number of ways for E_1 or E_2 to occur is $n_1 + n_2$.

For a finite set A, let $|A|$ denote the number of elements in A. Thus $|\{a, b, c\}| = 3$ while $|\emptyset| = 0$, where $\emptyset$ denotes the empty set. Using the language of sets, then the Addition Principle simply says that:

If A and B are finite sets such that $A \cap B = \emptyset$, then

$$|A \cup B| = |A| + |B|.$$

Clearly, the above result can be extended in a natural way to any finite number of pairwise disjoint finite sets.

(AP) If $A_1, A_2, \ldots, A_n$, $n \geq 2$, are finite sets which are pairwise disjoint; i.e., $A_i \cap A_j = \emptyset$ for all i, j with $1 \leq i < j \leq n$, then

$$|A_1 \cup A_2 \ldots \cup A_n| = |A_1| + |A_2| + \ldots + |A_n|,$$

or in a more concise form:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|.$$

Example 1.2

Find the number of squares contained in the 4 x 4 chessboard of Figure 1.1

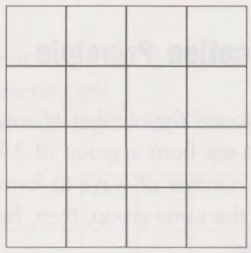


Figure 1.1

The squares in the chessboard can be divided into the following 4 sets:

A_1: the set of 1 x 1 squares,
A_2: the set of 2 x 2 squares,
A_3: the set of 3 x 3 squares, and
A_4: the set of 4 x 4 squares.

There are 16 1 x 1 squares. Thus $|A_1| = 16$. There are 9 2 x 2 squares. Thus $|A_2| = 9$.
Likewise,

$$|A_3| = 4 \text{ and } |A_4| = 1.$$

Clearly, the sets A_1, A_2, A_3, A_4 are pairwise disjoint and $A_1 \cup A_2 \cup A_3 \cup A_4$ is the set of the squares contained in the diagram of Figure 1.1. Thus by (AP), the desired number of squares is given by

$$\left|\bigcup_{i=1}^{4} A_i\right| = \sum_{i=1}^{4} |A_i| = 16 + 9 + 4 + 1 = 30.$$

Problem 1.1
Do the same problem as in Example 1.1 for 1 x 1, 2 x 2, 3 x 3 and 5 x 5 chessboards. Observe the patterns of your results. Find in general the number of squares contained in an n x n chessboard, where n ≥ 2.

Problem 1.2
Find the number of triangles contained in the configuration of Figure 1.2.

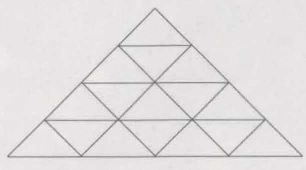


Figure 1.2

Problem 1.3

Find the number of squares contained in the configuration of Figure 1.3.

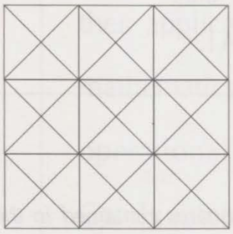


Figure 1.3

2. The Multiplication Principle

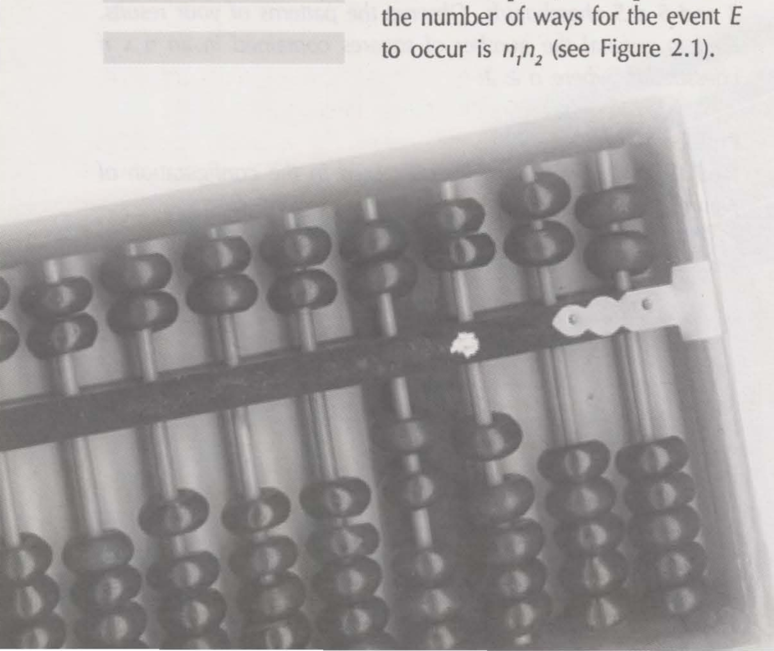
In Example 1.1, we found the number of ways to form a pair of students of the *same* sex from a group of 3 boys and 4 girls. If we wish to find the number of ways to form a pair of students of *different* sex from the same group, then, by listing all possible pairs as shown below:

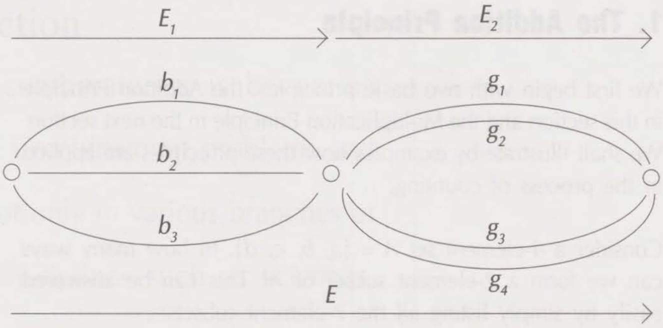
$$\{b_1, g_1\}, \{b_1, g_2\}, \{b_1, g_3\}, \{b_1, g_4\},$$
$$\{b_2, g_1\}, \{b_2, g_2\}, \{b_2, g_3\}, \{b_2, g_4\},$$
$$\{b_3, g_1\}, \{b_3, g_2\}, \{b_3, g_3\}, \{b_3, g_4\},$$

we get the answer '3 x 4' or '12'. In forming a pair of boy and girl, we may first select a boy and then select a girl. These two selections are independent. There are 3 ways for the first event to occur and 4 ways for the second event to occur. Thus the answer is '3 x 4'.

Very often, a given event may be split into ordered stages ((a boy) first and then (a girl)). In this case, the desired number of ways that the given event can occur is equal to the product of the numbers of ways in the respective stages. This way of counting is justified by the following:

The Multiplication Principle
Suppose that an event E can be split into two events E_1 and E_2 in ordered stages. If there are n_1 ways for E_1 to occur and n_2 ways for E_2 to occur, the number of ways for the event E to occur is $n_1 n_2$ (see Figure 2.1).





E : forming a pair of boy and girl
E_1: selecting a boy
E_2: selecting a girl

Figure 2.1

Given two sets A and B, let

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

Then the Multiplication Principle simply says that:
For any two finite set A and B,

$$|A \times B| = |A| \times |B|.$$

Likewise, the above result can be extended in a natural way when the event E is split into any finite number of ordered stages.

Example 2.1

There are four 2-digit binary sequences: 00, 01, 10, 11. There are eight 3-digit binary sequences: 000, 001, 010, 011, 100, 101, 110, 111. How many 6-digit binary sequences can we form?

The event of forming a 6-digit binary sequence can be split into 6 ordered stages as shown in Figure 2.2.

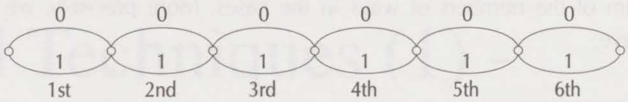


Figure 2.2

Thus by the Multiplication Principle, the desired number of sequences is $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$.

Let $A_1, A_2, \ldots, A_n$, $n > 2$, be n finite sets and let $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i$ for each $i = 1, 2, \ldots, n\}$.

Then the general form of the Multiplication Principle, using the language of sets, may be stated as follows:

(MP)
$$|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \times |A_2| \times \ldots \times |A_n|.$$

For instance, consider the problem discussed in Example 2.1, we have

$$A_1 = A_2 = \ldots = A_6 = \{0, 1\}.$$

The elements in $A_1 \times A_2 \times \ldots \times A_6$ can be identified with the

6-digit binary sequences in the following:

$$(1, 1, 0, 1, 0, 1) \longleftrightarrow 110101$$
$$(0, 1, 1, 1, 0, 0) \longleftrightarrow 011100$$

Thus the number of 6-digit binary sequences is equal to $|A_1 \times A_2 \times \ldots \times A_6|$, which by (MP), is equal to $|A_1| \times |A_2| \times \ldots \times |A_6| = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$.

We have seen in the preceding examples how (AP) or (MP) can be applied *individually*. These two principles become much more powerful tools if they are suitably combined.

Example 2.2
Figure 2.3 shows eight distinct points a_1, a_2, b_1, b_2, b_3, c_1, c_2, c_3 chosen from the sides of ΔABC. How many triangles can be formed using these points as the vertices?

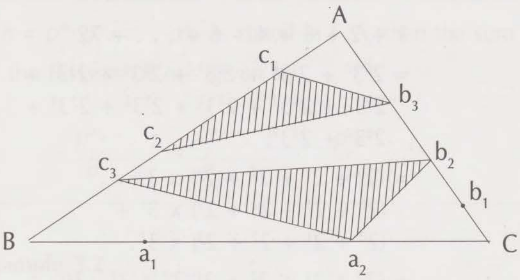


Figure 2.3

As shown in Figure 2.3, we notice that such triangles are of two types: (1) one vertex from each side (such as $a_2 b_2 c_3$), (2) two vertices from a side and one vertex from another side (such as $b_3 c_1 c_2$).

The number of triangles of types (1) is, by (MP), $2 \times 3 \times 3 = 18$.
The number of triangles of type (2) is, by (AP) and (MP),

$$\underset{\substack{\uparrow \\ \{a_1,\, a_2\}}}{1}(3 + 3) + \underset{\substack{\uparrow \\ \{b_1,\, b_2\} \\ \{b_1,\, b_3\} \\ \{b_2,\, b_3\}}}{3}(3 + 2) + \underset{\substack{\uparrow \\ \{c_1,\, c_2\} \\ \{c_1,\, c_3\} \\ \{c_2,\, c_3\}}}{3}(2 + 3) = 36.$$

Thus by (AP), the desired number of triangles is equal to $18 + 36$; i.e., 54.

Problem 2.1
There are 2 distinct terms in the expansion of $a(p + q)$:

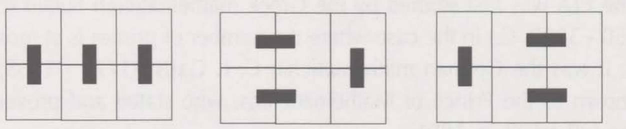
$$a(p + q) = ap + aq.$$

There are 4 distinct terms in the expansion of $(a + b)(p + q)$:

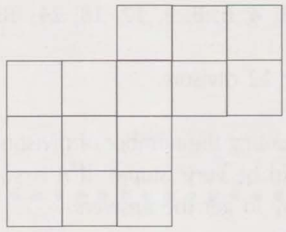
$$(a + b)(p + q) = ap + aq + bp + bq.$$

How many distinct terms are there in the expansion of
(i) $(a + b + c + d)(p + q + r + s + t)$,
(ii) $(x_1 + x_2 + \ldots + x_m)(y_1 + y_2 + \ldots + y_n)$, and
(iii) $(x_1 + x_2 + \ldots + x_m)(y_1 + y_2 + \ldots + y_n)(z_1 + z_2 + \ldots + z_k)$?

Problem 2.2
A 3×2 rectangle can be covered by 2×1 rectangles in 3 different ways as shown below:



In how many different ways can the following configuration be covered by 2×1 rectangles?
(See Singapore Math. Olympiad for Primary Schools, 1994)



3. Divisors of Natural Numbers

In this section, we shall study two counting problems related to the divisors of natural numbers. For convenience, we denote by

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

the set of all natural numbers.
Assume that $d, n \in \mathbb{N}$. We say that d is a *divisor* of n if when n is divided by d, the remainder is zero. Thus 3 is a divisor of 12, 5 is a divisor of 100, but 2 is not a divisor of 9.

Let $n \in \mathbb{N}$, $n \geq 2$. Clearly n has at least two divisors, namely 1 and n. How many divisors (inclusive of 1 and n) does n have? This is a type of problems that can often be found in mathematical competitions. We shall tackle this problem and see also how (MP) can be used in the solution.

To understand the solution, we first recall a special type of numbers, called prime numbers, and state an important result relating natural numbers and prime numbers.

A natural number p is said to be *prime* (or called a *prime*) if $p \geq 2$ and the only divisors of p are 1 and p. All prime numbers less than 100 are shown below:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

The primes are often referred to as building blocks of numbers because every natural number can always be expressed uniquely as a product of some primes. For example,

$$72 = 2^3 \times 3^2, \qquad 3146 = 2 \times 11^2 \times 13$$
$$1620 = 2^2 \times 3^4 \times 5, \qquad 12600 = 2^3 \times 3^2 \times 5^2 \times 7.$$

How to divide?
How can you divide a cake between two people so that each is satisfied that he/she has at least half the cake? Can you devise a procedure to share a cake between 3 people so that each is satisfied that he/she has at least one-third of the cake?

This fact is so basic and important to the study of numbers that it is called the Fundamental Theorem of Arithmetic (FTA).

> FTA. Every natural number $n \geq 2$ can be factorized as
>
> $$n = p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}$$
>
> for some primes $p_1, p_2, \ldots, p_k$ which are pairwise distinct and some natural numbers $m_1, m_2, \ldots, m_k$. Such a factorisation is unique if the order of primes is disregarded.

The FTA was first studied by the Greek mathematician Euclid (c. 450 - 380B. C.) in the case where the number of primes is at most 4. It was the German mathematician C. F. Gauss (1777 - 1855), known as the Prince of Mathematicians, who stated and proved the full result in 1801.

Let us now return to the problem of counting the number of divisors on n. How many divisors does '72' have? Since '72' is not a big number, we can get the answer simply by listing all divisors of 72:

$$1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 72.$$

Thus '72' has exactly 12 divisors.

The above way of counting the number of divisors of n by listing all divisors of n would be very 'stupid' if n is sufficiently large. Is there a simpler way to get the answer?

Let us look at the example when $n = 72$ again.
Observe that $72 = 2^3 \times 3^2$. Thus by FTA, a number is a divisor of '72' if and only if it is of the form $2^x \times 3^y$, where x, y are integers such that $0 \leq x \leq 3$ and $0 \leq y \leq 2$. Indeed, we have

$$\begin{aligned} 1 &= 2^0 \times 3^0, & 9 &= 2^0 \times 3^2, \\ 2 &= 2^1 \times 3^0, & 12 &= 2^2 \times 3^1 \\ 3 &= 2^0 \times 3^1, & 18 &= 2^1 \times 3^2, \\ 4 &= 2^2 \times 3^0, & 24 &= 2^3 \times 3^1, \\ 6 &= 2^1 \times 3^1, & 36 &= 2^2 \times 3^2, \\ 8 &= 2^3 \times 3^0, & 72 &= 2^3 \times 3^2 \end{aligned}$$

This observation suggests that the number of divisors of '72' is closely related to the numbers of choices of x and y. As a matter of fact, since x has 3 + 1 choices (i.e., 0, 1, 2, 3) and y has 2 + 1 choices (i.e., 0, 1, 2), it follows by (MP) that the number of integers of the form $2^x \times 3^y$ where $0 \leq x \leq 3$ and $0 \leq y \leq 2$, and hence the number of divisors of '72', is given by $(3 + 1)(2+1) = 12$, which agrees with the above listing.

Example 3.1
Find the number of divisors of 12600.

Observe that $12600 = 2^3 \times 3^2 \times 5^2 \times 7^1$.

Thus a number is a divisor of 12600 of and only if it is of the form

$$2^a \times 3^b \times 5^c \times 7^d$$

where a, b, c, d are integers

such that $0 \leq a \leq 3$, $0 \leq b \leq 2$, $0 \leq c \leq 2$ and $0 \leq d \leq 1$.

Since a has 3 + 1 choices, b and c each has 2 + 1 choices and d has 1 + 1 choices,

the number of divisors of 12600 is, by (MP),

$$(3 + 1)(2 + 1)(2 + 1)(1 + 1) = 72$$

Using a similar argument, it can be shown in general the following result.

> If $n = p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}$, as stated in FTA, then the number of divisors of n is given by
>
> $$(m_1 + 1)(m_2 + 1) \ldots (m_k + 1).$$

Consider another problem about divisors. If we sum up the 12 divisors of 72, we get

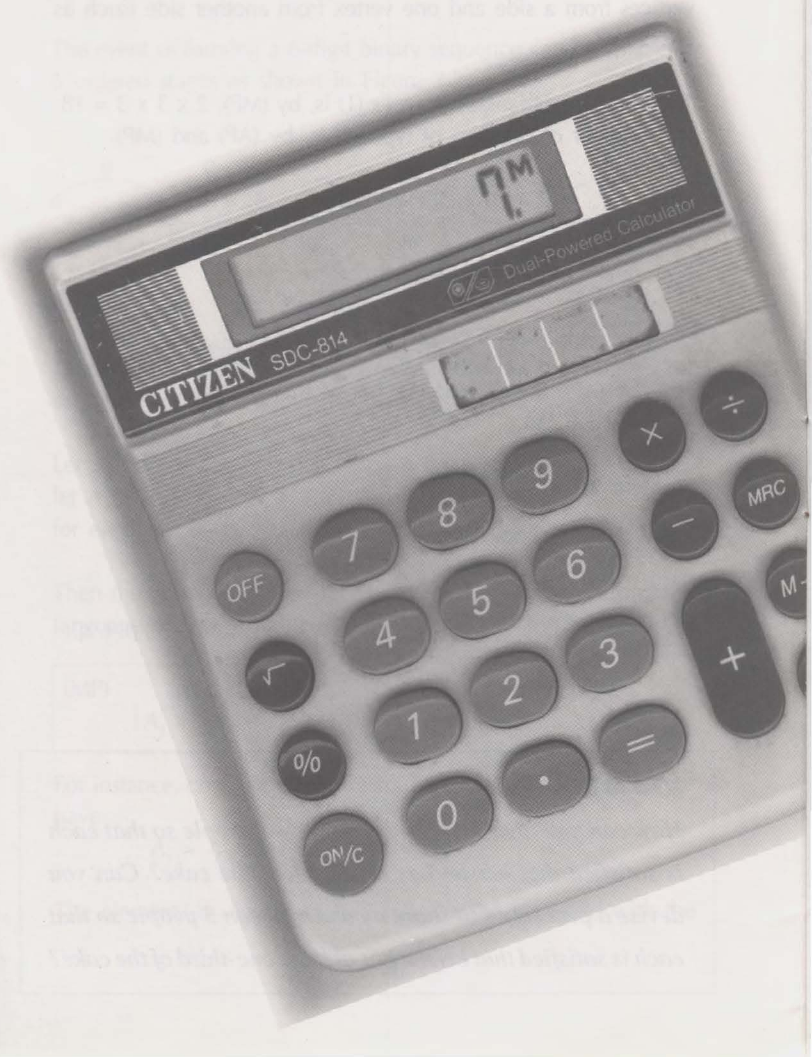
$$1 + 2 + 3 + 4 + 6 + 8 + 9 + 12 + 18 + 24 + 36 + 72 = 195.$$

The question is : can we get the sum without knowing the divisors of 72?

Let us do the above summation in the following way:

$$\begin{aligned} & 1 + 2 + 3 + 4 + 6 + \ldots + 72 \\ &= 2^0 3^0 + 2^1 3^0 + 2^0 3^1 + 2^2 3^0 + 2^1 3^1 + \\ &\quad 2^3 3^0 + 2^0 3^2 + 2^2 3^1 + 2^1 3^2 + 2^3 3^1 + \\ &\quad 2^2 3^2 + 2^3 3^2 \\ &= (2^0 + 2^1 + 2^2 + 2^3) \times 3^0 + \\ &\quad (2^0 + 2^1 + 2^2 + 2^3) \times 3^1 + \\ &\quad (2^0 + 2^1 + 2^2 + 2^3) \times 3^2 \\ &= (2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2). \end{aligned}$$

Observe that $2^0 + 2^1 + 2^2 + 2^3$ is a geometric series with 3 + 1 terms where 3 is the power of 2 in $72 = 2^3 \times 3^2$; and $3^0 + 3^1 + 3^2$ is a geometric series with 2 + 1 terms where 2 is the power of 3 in $72 = 2^3 \times 3^2$.



With the help of the following formula for the sum of a geometric series:

$$r^0 + r^1 + r^2 + \ldots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

we see that the sum of the divisors of 72 is

$$\begin{aligned}&(2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)\\ &= \frac{2^4 - 1}{2 - 1} \times \frac{3^3 - 1}{3 - 1}\\ &= 15 \times 13\\ &= 195\end{aligned}$$

which agrees with what we got before.
In general, we have the following result:

> If $n = p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}$ as stated in FTA, then the sum of the divisors of n is given by
>
> $$\frac{p_1^{m_1+1} - 1}{p_1 - 1} \times \frac{p_2^{m_2+1} - 1}{p_2 - 1} \ldots \frac{p_k^{m_k+1} - 1}{p_k - 1}.$$

Example 3.2
Find the sum of the divisors of 12600

Since $12600 = 2^3 \times 3^2 \times 5^2 \times 7^1$, the desired sum is equal to

$$\begin{aligned}&\frac{2^4 - 1}{2 - 1} \times \frac{3^3 - 1}{3 - 1} \times \frac{5^3 - 1}{5 - 1} \times \frac{7^2 - 1}{7 - 1}\\ &= 15 \times 13 \times 31 \times 8\\ &= 48360\end{aligned}$$

Problem 3.1
Find the number of divisors and the sum of the divisors of (i) 96, (ii) 1620.

Problem 3.2
We have shown that the number of divisors of 72 is 12. Are there any other 2-digit numbers which have exactly 12 divisors? Find out all such 2-digit numbers. M^2

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