You have learned many basic properties of plane geometrical figures such as the triangles, the parallelograms and the circles. There are however many interesting and fascinating elementary results which are not covered in our secondary school mathematics syllabus. In this Geometry Corner, we will introduce to you some of these results and hope that you would in the long run gain a better insight of plane geometry. You are encouraged to follow the proofs of these results, as it provides an understanding of the thinking process arising from the essential ideas.

We shall begin with a famous result known as the Steiner-Lehmus Theorem (S-L Theorem).

THE STEINER-LEHMUS THEOREM

by Hang Kim Hoo & Koh Khee Meng

The Steiner-Lehmus Theorem

Figure 1 shows a triangle ABC with two points D and E on AC and AB respectively satisfying the following condition:

(*) BD bisects $\angle ABC$ and CE bisects $\angle ACB$.

Consider the following problem.
If $AB = AC$, what can we say about the relation between BD and CE ?
More precisely, is it true that $BD = CE$?

To answer this question, we first note that $\angle ABC = \angle ACB$ (why?).
Consider ΔBCE and ΔCBD. We now claim that $\Delta BCE \equiv \Delta CBD$.
Observe that

$$\angle EBC = \angle DCB$$
$$BC = BC$$
$$\text{and} \quad \angle ECB = {}^1\!/_2\,\angle ACB = {}^1\!/_2\,\angle ABC = \angle DBC,$$
$$\text{i.e.,} \quad \angle ECB = \angle DBC.$$

Thus, $\Delta BCE \equiv \Delta CBD$ (*A.S.A.*), as claimed. It follows that $BD = CE$, answering the above question in the affirmative.

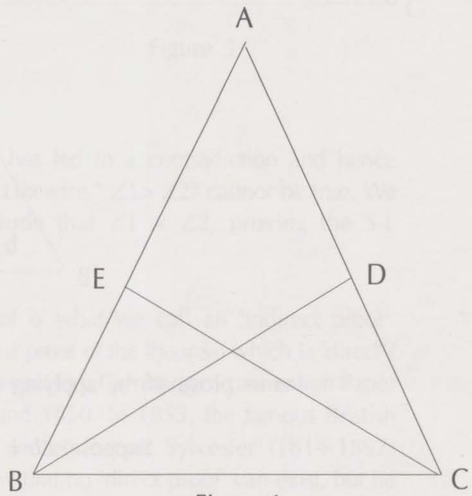

Figure 1

The above discussion shows that for a triangle ABC satisfying (*), if $AB = AC$, then $BD = CE$. What can be said about its *converse?* That is, for a triangle ABC satisfying (*), if $BD = CE$, is it always true that $AB = AC$? If you think of it for a while, you will then find this problem more challenging.

The above problem, which had baffled some famous mathematicians for quite a while, was first studied and claimed to be true by Lehmus of Berlin around 1840. Having found it very difficult to prove, he sought the help of another mathematician Sturm, who in turn mentioned it to a number of people including the great Swiss geometer, Jacob Steiner (1796 - 1863). Steiner soon confirmed that it is true but did not publish his proof until 1844. The result became known as the Steiner-Lehmus theorem. The full statement of the theorem is stated below.

Steiner-Lehmus Theorem

> Let ABC be a triangle with points D and E on AC and AB respectively such that BD bisects $\angle ABC$ and CE bisects $\angle ACB$. If $BD = CE$, then $AB = AC$.

The Method of Contradiction

Many proofs of the S-L Theorem have since been given, and we shall introduce to you one of them later. The method which we shall present is known as the method of contradiction. Since many of you may not be familiar with it, we give a simple example to illustrate it.

Suppose we have learned and accepted the following result.

(I) In ΔPQR of Figure 2, if $\angle q > \angle r$, then $PR > PQ$.

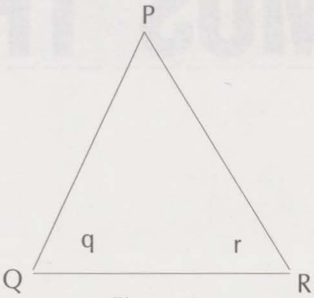

Figure 2

And now consider the following problem.

(II) Let ABC be a triangle of Figure 3. If $AB = AC$, show that $\angle b = \angle c$.

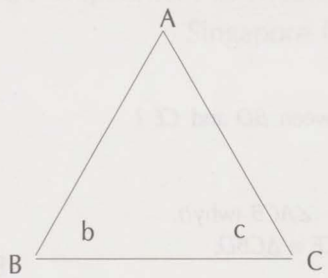

Figure 3

Let us prove (II) by applying (I) in the following 'indirect' way.

$$\text{Suppose } \angle b \neq \angle c \text{ ———— } (\#)$$

Then either $\angle b < \angle c$, $\angle b > \angle c$. Let us assume that $\angle b < \angle c$. By result (I), we have $AC < AB$. This, however, contradicts our given assumption that $AC = AB$. Thus our supposition (#) cannot hold, and we conclude that $\angle b = \angle c$, proving (II).

Note that in the above discussion, we did not directly prove that $\angle b = \angle c$, but concluded that $\angle b = \angle c$ from the fact that " $\angle b \neq \angle c$ " is not possible. Such a method of proof is what mathematicians term as the method of contradiction. This method of proof is extremely powerful, and has been used by mathematicians to prove many results.

Proof of the S-L Theorem

The proof of the S-L Theorem we are going to present is by contradiction. It is an elegant proof, and as far as we know, it is one of the shortest ones among the existing proofs.

Before we proceed, we state the following simple fact that will be used later.

(III) Let PQR and XYZ be two triangles of Figure 4 such that $PQ = XY$ and $QR = YZ$. If $\angle q > \angle y$, then $PR > XZ$.

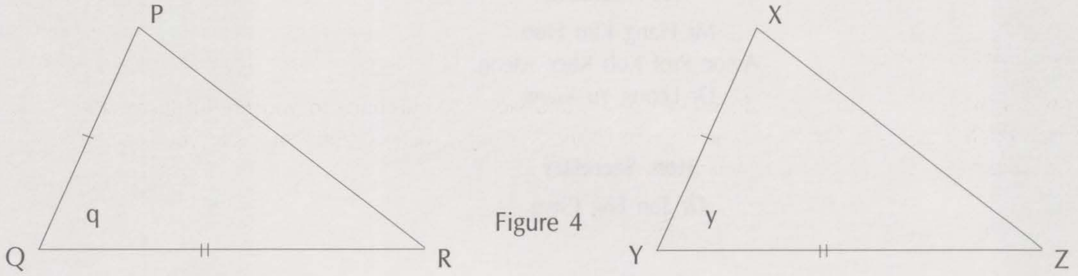

Figure 4

We are now ready to prove the S-L Theorem.

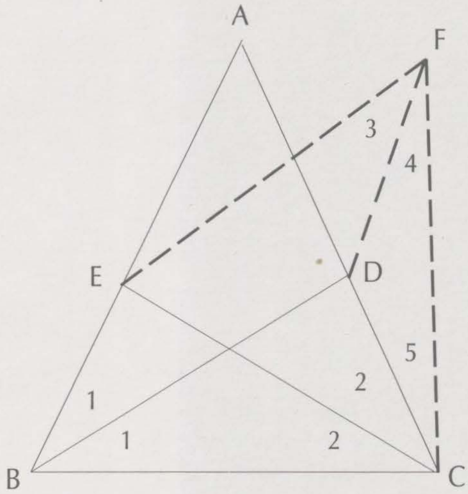

Figure 5

Proof

Since BD and CE are angle bisectors, for easy reference, let $\angle ABD = \angle CBD = \angle 1$ and $\angle ACE = \angle BCE = \angle 2$ as shown in Figure 5.

To prove that $AB = AC$, it suffices to prove that $\angle 1 = \angle 2$.

Suppose $\angle 1 \neq \angle 2$. Then either $\angle 1 < \angle 2$ or $\angle 1 > \angle 2$. Let us assume that

$$\angle 1 < \angle 2 \text{ ——— } (1)$$

Consider ΔBCD and ΔCBE. Since $BD = CE$ (given), $BC = CB$ and $\angle 1 < \angle 2$, by result (III),

$$CD < BE \text{ ——— } (2)$$

Next, let F be a point on the plane of ABC such that $BDFE$ is a parallelogram. Let $\angle DFE = \angle 3$, $\angle DFC = \angle 4$ and $\angle DCF = \angle 5$.

Since $EF = BD$ (why?)
$= CE$ (given),
we have $\angle CFE = \angle ECF$,
i.e. $\angle 3 + \angle 4 = \angle 2 + \angle 5$
But $\angle 3 = \angle 1$ (why?)
Thus, $\angle 1 + \angle 4 = \angle 2 + \angle 5$
Now, by (1), we have
$\angle 4 > \angle 5$,
which implies that $CD > DF$.
Since $DF = BE$ (why?)
we have $CD > BE$
which, however, contradicts (2).

Supposition (1) has led to a contradiction and hence cannot be true. Likewise "$\angle 1 > \angle 2$" cannot be true. We therefore conclude that $\angle 1 = \angle 2$, proving the S-L Theorem.

The above proof is what we call an 'indirect proof'. Does there exist a proof of the theorem which is 'direct'? This problem was set in a Cambridge Examination Paper in England around 1850. In 1853, the famous Bristish mathematician James Joseph Sylvester (1814-1897) intended to show that no 'direct proof' can exist, but he was not very successful. Since then, there have been a number of 'direct proofs' published, but strictly speaking no one is 'direct' as they require some other results which have not been proved directly. Those who like to read more about this may refer to the article "The equal internal bisectors theorem" by J. A. McBride published in the Proc. Edinburgh Maths. Society, Edinburgh Maths. Notes 33 (1943), 1-13. M²

Mr Hang Kim Hoo obtained his BSc with Honours in Mathematics from NUS and MEd from NTU. His research interest lies in the teaching of Geometry. He has many years of experience in teaching mathematics at secondary schools and is currently a Specialist Inspector for Mathematics at the Ministry of Education. He has been a member of the International Mathematics Olympiad Training Committee since 1990.