A counterexample to the infinite version of a

theorem of van der Waerden

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The theorem of van der Waerden on arithmetic progressions states that if the set of natural numbers $N = \{1, 2, 3, ..., \}$ is divided in any manner into two sets A and B (for example A contains even numbers and B contains odd numbers) then one of these two sets contains finite arithmetic progressions of arbitrary number of terms. [**1**, chapter 1].

This result was first conjectured by I. Schur in 1906 in connection with his work in quadratic residues and proved by B. L. van der Waerden in 1926. We refer the readers to the paper [**2**] for an interesting account of van der Waerden's discovery.

Although it is well-known that the infinite version of this theorem is false, counterexamples are not readily available in the literature. In this note we shall give a simple construction of such a counterexample. In other words we shall find a partition of N into two parts A and B such that neither A nor B contains an infinite arithmetic progression.

Each infinite arithmetic progression of natural numbers is of the form $a, a + d, a + 2d, a + 3d, ...$ and therefore is uniquely determined by its first term a and its common difference d and so it can be identified with the ordered pair $(a, d) \in N \times N$.

Since $N \times N$ is countable (see the box) we can order it as the set $\{(a_1, b_1), (a_2, b_2), ..., \}$.
Now we define two sequences $\{x_n\}$ and $\{y_n\}$ by $y_0 = 1$, and $x_n = a_n + y_{n-1}b_n$, $y_n = a_n + x_nb_n$ for each $n \in N$.

It is clear from this definition that we have the following property

$$(*)\ x_1 < y_1 < x_2 < y_2 < x_3 < y_3 < ...$$

Next we divide N into two parts A and B where

$$A = \{x_n : n \in N\} \text{ and } B = \{m \in N: m \notin A\}.$$

We claim that no infinite arithmetic progression can be found in either one of these sets. To see this let us take an arbitrary infinite arithmetic progression which we shall identify it with some $(a_k, b_k) \in N \times N$. Therefore this arithmetic progression is

$$a_k,\ a_k + b_k,\ a_k + 2b_k,\ a_k + 3b_k,\ ...$$

We observe that by our definition $x_k = a_k + y_{k-1}b_k$ and $y_k = a_k + x_kb_k$ are both terms of this arithmetic progression but $x_k \in A$ by the definition of A and $y_k \notin A$ because by (*) y_k is different from x_i, for all $i = 1, 2, 3, ...$
Therefore neither A nor B can contain this entire arithmetic progression. M^2

References

[1] *A. Y. Khinchin*: **Three Pearls of Number Theory.**

[2] *B. L. van der Waerden*: **How the proof of Baudet's conjecture was found.**

A set S is said to be countable if there exists a bijective map from S to the set of natural number N. This is as good as saying that the elements of a countable set can be listed in a certain order as a first element, a second element, a third element, and so on. To see that $N \times N$ is countable, it is straight forward to check that the map which maps the element (m, n) in $N \times N$ to the element $m + p$ in N where $2p = (m + n - 2)(m + n - 1)$ is a bijective map. In this case, the elements of $N \times N$ are arranged in the order (1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), ...

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