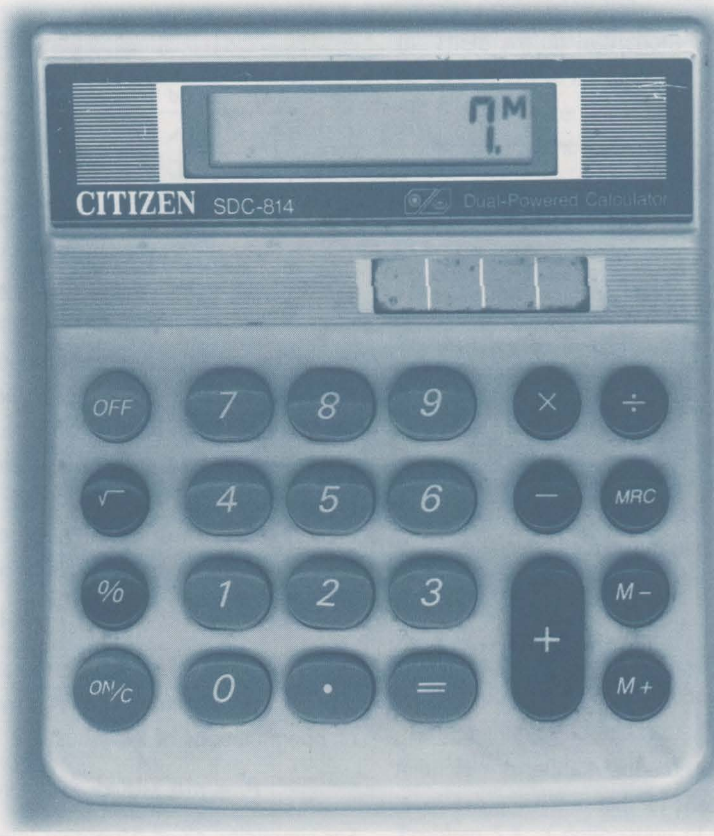
COUNTING
- Its Principles and Techniques (2) -

by K M Koh and B P Tan



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4. Subsets and Arrangements.

Given a set S of 10 objects, how many 3-element subsets of S are there? If, further, the 3 elements chosen are to be arranged in a row, where the ordering counts, how many ways can this be done? In this article, our attention will be focused on the aboved two basic counting problems. We shall see how the Multiplication Principle (MP) that we learned in Part (1) of the article can be used to solve the above problems, and how (MP) can be incorporated with the Addition Principle (AP) to enable us to solve more complicated problems.

Consider the 4-element set $\mathbb{N}_4 = \{1, 2, 3, 4\}$. How many subsets of $\mathbb{N}_4$ are there? This question can be answered readily by listing all the subsets of $\mathbb{N}_4$. Table 1 displays all the r-element subsets of $\mathbb{N}_4$, where $r = 0, 1, 2, 3, 4$. The numbers of r-element subsets (0 r 4) are recorded on the right hand column of the table.

	Subsets of N_4	number
0-element	$\emptyset$	1
1-element	{1}, {2}, {3}, {4}	4
2-element	{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}	6
3-element	{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}	4
4-element	{1, 2, 3, 4}	1

Table 1.

Numbers of this kind (i.e., 1, 4, 6, 4, 1) are so useful and important that mathematicians had introduced special symbols to denote them.

Given a positive integer n, let

$$\mathbb{N}_n = \{1, 2, \ldots, n\}.$$

For $r = 0, 1, \ldots, n$, let $\binom{n}{r}$ denote the number of r-element subsets of $\mathbb{N}_n$. Thus, Table 1 tells us that

$$\binom{4}{0} = 1,\ \binom{4}{1} = 4,\ \binom{4}{2} = 6,\ \binom{4}{3} = 4 \text{ and } \binom{4}{4} = 1.$$

The symbol $\binom{n}{r}$ is read 'n choose r'. Some other symbols for this quantity include C^n_r and ${}^nC_r$.

What is the value of $\binom{5}{2}$, the number of 2-element subsets of $\mathbb{N}_5 = \{1, 2, 3, 4, 5\}$? By listing all the 2-element subsets of $\mathbb{N}_5$ as shown below:

{1, 2}, {1, 3}, {1, 4}, {1, 5}, {2, 3}, {2, 4},
{2, 5}, {3, 4}, {3, 5}, {4, 5},

we see that there are '10' in number. Thus, by definition,

$$\binom{5}{2} = 10.$$

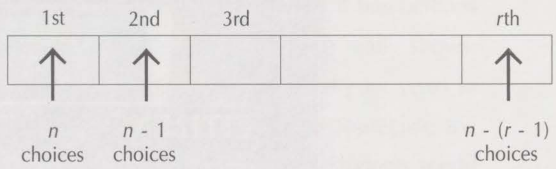
When n is large, it would be tedious to list all the r-element subsets of $\mathbb{N}_n$ just to determine what the value of $\binom{n}{r}$ is. Is there a more economical way to find $\binom{n}{r}$?

Before answering this question, let us consider a different but related problem. How many ways are there to arrange any two elements of $\mathbb{N}_4 = \{1, 2, 3, 4\}$ in a row? All such arrangements are shown below:

12, 13, 14, 21, 23, 24, 31, 32, 34, 41, 42, 43.

Thus there are '12' ways to do so. Indeed, we can get the answer easily without listing all such arrangements. We observe that there are '4' ways (choosing 1, 2, 3 or 4) to fill the 1st position, and '3' ways to fill the 2nd position. Thus, by (MP), the desired number of ways is $4 \cdot 3 = 12$, which agrees with what we have obtained.

In general, how many ways are there to arrange any r elements of $\mathbb{N}_n$, where 0 r n, in a row?



Consider the r spaces shown in the above diagram. We wish to choose r elements from $\{1, 2, \ldots, n\}$ to fill the r spaces, where the orderings of elements count. There are n choices for the 1st space. After fixing one in the 1st space, there are $n - 1$ choices for the 2nd space. After fixing one in the 2nd space, there are $n - 2$ choices for the 3rd space, and so on. After fixing one in $(r - 1)$th space, there are $n - (r - 1)$ choices for the rth space. Thus, by (MP), the number of ways to arrange any r elements from $\mathbb{N}_n$ in a row is given by

$$n(n - 1)(n - 2)\ldots(n - (r - 1)).$$

For convenience, let us call an arrangement of any r elements from $\mathbb{N}_n$ an *r-permutation* of $\mathbb{N}_n$, and denote by P^n_r the number of r-permutations of $\mathbb{N}_n$. Thus, we have

$$P^n_r = n(n - 1)(n - 2)\ldots(n - r + 1) \qquad (1)$$

For simplicity, given a positive integer n, define $n!$ to be the product of the n consecutive integers $n, n - 1, \ldots, 3, 2, 1$; that is,

$$n! = n(n - 1) \ldots 3 \cdot 2 \cdot 1.$$

Thus $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. The symbol '$n!$' is read '$n$ factorial'. By convention, we define $0! = 1$.

Using the 'factorial' notation, we now have

$$\begin{aligned} P^n_r &= n(n - 1)\ldots(n - r + 1) \\ &= \frac{n(n - 1)\ldots(n - r + 1)(n - r)(n - r - 1)\ldots 2 \cdot 1}{(n - r)(n - r - 1)\ldots 2 \cdot 1} \\ &= \frac{n!}{(n - r)!} \end{aligned}$$

That is,

$$\boxed{P^n_r = \frac{n!}{(n - r)!}} \qquad (2)$$

When $n = 4$ and $r = 2$, we obtain

$$P^4_2 = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{4.3.2.1}{2.1} = 4 \cdot 3 = 12,$$

which agrees with what we found before.

Consider two extreme cases when $r = 0$ and $r = n$ respectively. When r = 0, by (2),

$$P^n_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1.$$

(How could this be explained?) When $r = n$, an r-permutation of $\mathbb{N}_n$ is simply called a *permutation* of $\mathbb{N}_n$. Thus, by (2) and that $0! = 1$, the number of permutations of $\mathbb{N}_n$ is given by

$$P^n_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n! \qquad (3)$$

Problem 4.1. Show that for $1 \quad r \quad n$,

(i) $P^{n+1}_r = P^n_r + rP^n_{r-1}$;

(ii) $P^{n+1}_r = r! + r(P^n_{r-1} + P^{n-1}_{r-1} + \ldots + P^r_{r-1})$.

We shall now return to the problem of determining the value of $\binom{n}{r}$. The number P^n_r of r-permutations of $\mathbb{N}_n$ is given by (2). Let us count this number in a different way. To obtain an r-permutation of $\mathbb{N}_n$, we may proceed in the following manner: first select an r-element subset of $\mathbb{N}_n$, and then arrange the r elements of the subset in a row. The number of ways for the first step is, by definition, $\binom{n}{r}$, while that for the second step is, by (3), $r!$. Thus, by (MP),

$$P^n_r = \binom{n}{r} \cdot (r!).$$

Hence, by (2),

$$\boxed{\binom{n}{r} = \frac{P^n_r}{r!} = \frac{n!}{r!(n-r)!}.} \qquad (4)$$

When $n = 5$ and $r = 2$, we have, by (4),

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = 10,$$

which agrees with what we obtained before. Note that, in particular, we have $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$. By (4), we can now compute the value of $\binom{n}{r}$ without listing all the r-element subsets of $\mathbb{N}_n$.

We define P^n_r (respectively, $\binom{n}{r}$) as the number of r-permutations (respectively, r-element subsets) of $\mathbb{N}_n = \{1, 2, \ldots, n\}$. Actually, P^n_r (respectively, $\binom{n}{r}$) can be defined as the number of r-permutations (respectively, r-element subsets) of any n-element set S, since it is the number but not the nature of the elements in the set that counts. Any r-element subset of S is also called an *r-combination* of S. Thus, the quantity $\binom{n}{r}$ is the number of r-combinations of S.

Problem 4.2. Show that

(i) $\binom{n}{r} = \binom{n}{n-r}$, where $0 \quad r \quad n$;

(ii) $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, where $1 \quad r \quad n-1$.

5. Applications.

Having introduced the concepts of r-permutations and r-combinations of an n-element set, and deriving the formulae for P^n_r and $\binom{n}{r}$, we shall now give some examples to illustrate how these can be applied.

Example 5.1.

There are 4 girls and 5 boys in a class, which include two particular boys A and B, and one particular girl G. Find the number of ways to arrange them in a row in each of the following cases:

(i) There are no restrictions;

(ii) A and B are adjacent;

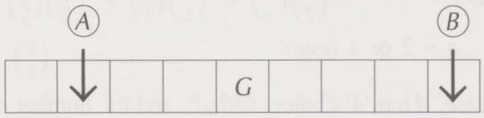
(iii) G is at the centre, A at her left (need not be adjacent) and B at her right.

Solution.

(i) This is the number of permutations of the 9 children. The answer is 9!.

(ii) Treat $\{A, B\}$ as a single entity. The number of ways to arrange the remaining 7 children together with this entity is $(7 + 1)!$. But A and B can permute themselves in '2' ways. Thus the desired number of ways is, by (MP), $2 \cdot 8!$.

(iii)



As shown above, A has '4' choices and B also has '4' choices. The remaining 6 children can be placed in 6! ways. By (MP), the desired number of ways is $4 \cdot 4 \cdot 6!$.

Problem 5.1. (Continuation of Example 5.1)

(iv) A and B are at the two ends;

(v) G is at the centre and adjacent to A and B;

(vi) A, B, and G form a single block (i.e., there is no other child between any two of them);

(vii) All girls form a single block;

(viii) All girls form a single block and all boys form a single block;

(ix) No two of A, B and G are adjacent;

(x) All boys form a single block and G is adjacent to A;

(xi) Boys and girls alternate;

(xii) G is between A and B (need not be adjacent).

Example 5.2.

Find the number of integers between 2000 and 5000 in which no digit is repeated in each of the following cases:

(i) There are no additional restrictions;
(ii) The integers are even.

Solution. Let *abcd* be a required integer.

(i) As shown in the diagram below, *a* has 3 choices (i.e., 2, 3, or 4), say $a = 2$.

a	b	c	d

$\uparrow$

{2, 3, 4}

Since no digit is repeated, a way of forming '*bcd*' corresponds to a 3-permutaion from the 9-element set {0, 1, 3, 4, . . . , 9}. Thus the required number of integers is

$$3 \cdot P^9_3.$$

(ii) Again, $a = 2$, 3, or 4. We divide the problem into two cases.

Case (1)

$a = 3$ (odd) $\boxed{3 \mid b \mid c \mid d}$

In this case, d has '5' choices (i.e., 0, 2, 4, 6 or 8), say $d = 4$. Then a way of forming '*bc*' is a 2-permutation from the 8-element set {0, 1, 2, 5, 6, 7, 8, 9}. Thus the required number of integers is $5.P^8_2$.

Case (2)

$a = 2$ or 4 (even)

In this case, *d* has '4' choices (Why?), and the number of ways to form '*bc*' is P^8_2. The required number of integers is $2.4.P^8_2$.

By (AP), the desired number of integers is

$$5.P^8_2 + 2.4.P^8_2 = 13.P^8_2.$$

Problem 5.2. (Continuation of Example 5.2)

(iii) The integers are odd;
(iv) The integers are divisible by 5;
(v) The integers are greater than 2345.

Example 5.3.

At a Japan-Singapore conference held in Singapore, there are 17 participants from the two countries. Among them, 9 are Japanese.

In how many ways can a 7-member committee be formed from these participants in each of the following cases:

(i) there are no restrictions?
(ii) there is no Singaporean in the committee?
(iii) there is exactly one Singaporean in the committee?
(iv) the committee consists of Singaporeans?
(v) there are at most three Singaporeans in the committee?

Solution.

(i) This is the number of 7-element subsets of a 17-element set. By definition, the desired number is $\binom{17}{7}$.

(ii) This is the number of ways to form a 7-member committee from the 9 Japanese. Thus the desired number is $\binom{9}{7}$.

(iii) We first select a member from the 8 Singaporeans and then select the remaining 6 from the 9 Japanese. By (MP), the desired number is $\binom{8}{1}\binom{9}{6} = 8\binom{9}{6}$.

(iv) Obviously, the desired number is $\binom{8}{7} = 8$.

(v) There are 4 cases; namely, *r* Singaporeans, where $r = 0$, 1, 2, 3. Thus, by (AP), the desired number is

$$\binom{9}{7} + \binom{8}{1}\binom{9}{6} + \binom{8}{2}\binom{9}{5} + \binom{8}{3}\binom{9}{4}.$$

Example 5.4.

As shown in Example 2.1 (Part (1)), the number of 6-digit binary sequences is 2^6. How many of them contain exactly two 0's (e.g., 001111, 101101, . . .)?

Solution

Forming a 6-digit binary sequence with two 0's is the same as choosing two spaces from the following 6 spaces into which the two 0's are put (the rest are then occupied by 1's) as shown below

(1)	(2)	(3)	(4)	(5)	(6)
	0 $\uparrow$		0 $\uparrow$		
1	0	1	0	1	1

Thus the number of such binary sequences is $\binom{6}{2}$.

Example 5.5.

Figure 5.1 shows 10 distinct points on the circumference of a circle.

(i) How many chords of the circle are there formed by these points?
(ii) If no 3 of the chords are concurrent within the circle, how many points of intersection of these chords are there within the circle?

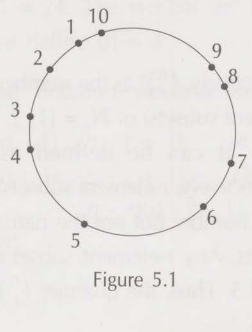
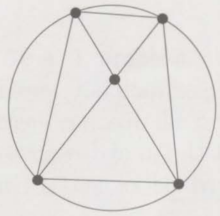


Figure 5.1

Solution

(i) Every chord joins two of the ten points, and every two of the ten points determine a unique chord. Thus the required number of chords is $\binom{10}{2}$.

(ii)



Every point of intersection of two chords corresponds to four of the ten points, and every four of the ten points determine a point of intersection. Thus the required number of points of intersection is $\binom{10}{4}$.

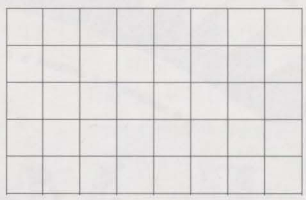
Problem 5.3.

In how many ways can a committee of 5 be formed from a group of 11 people consisting of 4 teachers and 7 students if

(i) the committee must include exactly 2 teachers?
(ii) the committee must include at least 2 teachers?
(iii) a particular teacher and a particular student are both in the committee?

Problem 5.4.

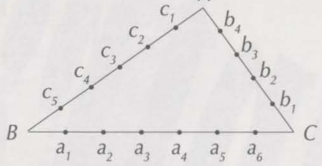
How many rectangles are there in the following 5 x 8 rectangular grid?



Problem 5.5.

The following figure shows 15 distinct points chosen on the sides of ABC.

(i) How many triangles can be formed from these points?
(ii) How many line segments are there joining any 2 points on different sides?
(iii) If no 3 of these line segments are concurrent, find the number of points of intersection of these line segments within ABC.



Problem 5.6.

The number 5 can be expressed as a sum of 3 natural numbers, taking order into account, in 6 ways;

$$\begin{aligned} 5 &= 1 + 1 + 3 = 1 + 3 + 1 \\ &= 3 + 1 + 1 = 1 + 2 + 2 \\ &= 2 + 1 + 2 = 2 + 2 + 1. \end{aligned}$$

In how many ways can 11 be written as a sum of 5 natural numbers, taking order into account?

Problem 5.7.

Four people A, B, C and D can be paired off in the following three different ways:

(1) $\{\{A, B\}, \{C, D\}\}$,

(2) $\{\{A, C\}, \{B, D\}\}$,

(3) $\{\{A, D\}, \{B, C\}\}$.

In how many ways can 10 people be paired off? M²

Answers

Problem 5.1.

(iv) 2.7! (v) 2.6! (vi) 3!7! (vii) 4!6! (viii) 4!5!2!
(ix) 6!7.6.5 (x) 2.4!4! (xi) 4!5! (xii) 2.4.5.6.7.8.9

Problem 5.2.

(iii) $2.5P_2^8 + 4.P_2^8 = 14.P_2^8$

(iv) $2.3.P_2^8 = 6.P_2^8$

(v) $2.P_3^9 + 6.P_2^8 + 5.P_1^7 + 4$ or $3.P_3^9 - 2.P_2^8 - 2.P_1^7 - 3$

Problem 5.3.

(i) $\binom{4}{2}\binom{7}{3}$

(ii) $\binom{4}{2}\binom{7}{3} + \binom{4}{3}\binom{7}{2} + \binom{4}{4}\binom{7}{1}$

(iii) $\binom{9}{3}$

Problem 5.4.

$\binom{6}{2}\binom{9}{2}$

Problem 5.5.

(i) $6{\cdot}4{\cdot}5 + \binom{6}{2}9 + \binom{4}{2}11 + \binom{5}{2}10$

(ii) $6{\cdot}4 + 4{\cdot}5 + 5{\cdot}6$

(iii) $\binom{6}{2}\binom{4}{2} + \binom{4}{2}\binom{5}{2} + \binom{5}{2}\binom{6}{2}$
$+ \binom{6}{2}{\cdot}4{\cdot}5 + \binom{4}{2}{\cdot}5{\cdot}6 + \binom{5}{2}{\cdot}6{\cdot}4$

Problem 5.6.

$\binom{10}{4}$

Problem 5.7

9.7.5.3.1 or $\dfrac{\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{5!}$