Mathematics *is a* Language

by Leong Yu Kiang

English Tamil CHINESE Singlish

It's English... It's Singlish...No, it's Mathematish!

Mathematics, a language? Like English, Chinese, Malay and Tamil? Certainly, mathematics has its own notations, symbols and even syntax. It is possible, in principle, to write a piece of mathematics within a completely closed system with its own symbolism independently of any human language. If this had been carried out (which, fortunately, had never been seriously attempted and enforced by decree or edict other than in some scholarly and isolated works on logic and the foundations of mathematics), then mathematics would indeed be a language of its own, at least in written form. The true, or at least historical, state of affairs is that mathematics is written in a mixture of human language and a unique symbolism.

When I say that mathematics is a language, I do not mean the visual or even oral aspect of it. That is why it is still being written in English, Chinese, Japanese, Russian or whatever language you think in. The presence of a human linguistic element is really irrelevant. Just imagine a universal linguist (AUL for short) who is able to read any written human language on earth. Given a proof of a mathematical statement, would AUL be able to understand it? Would the mathematical statement itself make any sense to her (after all, females are generally acknowledged to be better in languages than males)? More importantly, would she be able to tell whether the proof is correct? If she could understand the proof, we would be inclined to believe that she has been mathematically trained. If she could improve on the proof or perhaps find an error on the proof and rectify it, we would believe that she is a mathematician.

Mathematicians are well-known, if not notorious, for using commonplace words to represent their own concepts. In a book on algebra, you will find words like "rings", "fields" and "groups" and you can be sure that they have nothing to do with diamonds, sports and meetings. If you happen to come across a volume on "the theory of group representations" in the library, hesitate to think that it is an academic study of Singaporean GRCs (Group Representation Constituencies). It is also unlikely that a book with the title "An introduction to group theory" will unlock the secrets behind making friends and influencing people. Neither will one entitled "Nonlinear modelling" set you on a path to international high fashion.

You will also have the feeling that numbers can get temperamental because they can be rational or, even worse, irrational. Words which used to be as clear as daylight soon diffuse into a fog-laden twilight when they are spoken by your mathematics tutor. Whatever happened to good old-fashioned transformations, images, ranges, ranks and signatures? You also wonder whether you have entered the realm of science fiction as you are confronted with the "annihilator of a space of functions". (Fortunately, we have yet to come face to face with the "terminator of a vector space"!) And if you venture far enough into the realm of "chaos" and "solutions by radicals", you realise that this is not for the conservative-minded.

While the importance of mathematics is understood well enough to make "elementary" mathematics compulsory in the school curriculum up to secondary level, the abstruse nature of the mathematical language at the higher level seems to have relegated the subject to a position reserved only for hard-core specialists or for the purpose of "teaching those who will teach mathematics to those who will teach mathematics to . . . ". All others who venture in are politely reminded that they do so at their own risks!

Symbols galore... Prose unintelligible...

Here is an example of mathematics expressed in one of the most esoteric form which is absolutely unreadable as well as unintelligible to the uninitiated.

$\vdash . * 54 \cdot 26 . \supset \vdash : . \alpha = \iota' x . \beta = \iota' y . \supset : \alpha \cup \beta \in 2 . \equiv . x \quad y .$

$[*51 \cdot 231] \qquad \equiv . \iota' x \cap \iota' y = \Lambda .$

$[*13 \cdot 12] \qquad \equiv . \alpha \cap \beta = \Lambda \quad (1)$

$\vdash . (1) . * 11 \cdot 11 \cdot 35 . \supset$

$\qquad \vdash : . (\exists x, y) . \alpha = \iota' x . \beta = \iota' y . \supset : \alpha \cup \beta \in 2 . \equiv \alpha \cap \beta = \Lambda \quad (2)$

$\vdash . (2) . * 11 \cdot 54 . * 52 \cdot 1 . \supset \vdash . \text{Prop}$

Each symbol carries its own meaning and is independent of the language of the reader. Verbalizing the sequence of symbols involves a translation from the abstract notation to the language of the reader. No recognisable word of English (except for one) or of any other human language is discernible and yet the initiated reader is able to make sense out of it. This illustrates clearly the universality of mathematics as a language. This example is taken from the book *Principia Mathematica, Volume 1* on the foundations of arithmetic by the logicians Bertrand Russell (1872 - 1970) and Alfred North Whitehead (1861-1947), and it is supposed to be a proof from first principles of the result "1 + 1 = 2."

Not all mathematics is devoid of ordinary language. Much of modern mathematics is expressed in prose interspersed with abstract symbols. The following is from a talk by the group theorist Otto Kegel given at a group theory conference in 1987.

> Fix a prime p. We shall consider the question whether and when the maximal p-subgroups of the locally finite group G are all conjugates in G. If this is so, we shall say that G satisfies the *Sylow Theorem for the prime p.* Thus we suppress the arithmetic part of the classical Sylow Theorem for finite groups. We shall say that G satisfies the *strong Sylow Theorem for the prime p* if every subgroup of G satisfies the Sylow Theorem for the prime p. In general, the validity of the Sylow Theorem for the prime p does not imply the validity of the strong Sylow Theorem for p, as we shall see. But

clearly a necessary condition for the set $M_p(G)$ of all maximal p-subgroups of G to be one orbit under conjugation is the cardinal inequality $|M_p(G)| < |G|$.

The contrast between this example and the preceding one is striking. The whole passage can be articulated by the uninitiated who will nevertheless be none the wiser at the end of the articulation. Not just because certain words like conjugates, Sylow Theorem, etc., may not be in his vocabulary. Even if these words were defined precisely to him, comprehension breaks down at the logical level. The logic behind the statements are only accessible to the expert in the field.

A language with its own thought processes...

Probably because it is the utilitarian aspect of mathematics that is first taught, most people equate mathematics with computing. If mathematics is just a series of computations, it would indeed be a routine matter to verify them. Our first exposure to geometry as postulated by the ancient Greek geometer Euclid (around 3rd Century B.C.) quickly tells us that this is not so. We soon become aware that the language of mathematics has its own syntax (such as "If . . ., then . . .", "There exists some . . .," "Proof by contradiction") with a built-in thought process. In principle, each mathematical statement can be deduced from first principles, i.e. from the axioms or assumptions that are accepted as true. However, because of the accumulative nature of the results, going back to first principles will be prohibitive in terms of time and space. (For instance, the example given earlier of Russell and Whitehead's "proof" that "1 + 1 = 2" occurs on page 362 of their logical treatise.)

So, a mathematical theory is developed hierarchically in such a way that at a higher level, the body of results at lower levels are condensed or sublimated into nutshells which are directly accessed in the thought process of the expert. The assimilation of these nutshells must be thorough for a higher level of understanding, and this demands the mastery of those sublimated secondary thought processes. To use a simple analogy from the game of chess: it is not just perceiving mate in four or five moves, it is more like mate in ten moves or more.

To further appreciate the difficulty and depth of the secondary thought processes in understanding mathematics, let us again look at the language of chess which is familiar to any serious chess player. A typical example of chess literature is the following excerpt from an analysis (annotation or commentary) by Tony Dempsey in the *Singapore Chess Digest.*

> **22. Rc5!!** A very attractive sacrifice of the exchange. If declined with, for example, 22. . . . Qd8, 23R. R x c4 recovers the pawn with attacking chances hardly diminished and a clear advantage to White.
> **22. . . . B x c5 23. d x c5** with this capture both White's prelates are firing unopposed.
> **23. . . . f6 24. Q x c4 + Rf7?** Haba fails to find the most stubborn defence. After 24. . . . Qf7 25. Qh4, both 25. . . . g6 and 25. . . . h6 fail against 26. Qg3, hitting b8 and threatening Bb3, e.g. 26. . . . Ne7 27. Bb3 Nd5 28. Rd1 Rbd8 29. Qf3. So here Black has to be content with 25. . . . f5 though he remains clearly worse after 26. B x f5 Q x f5 27. Qg3.

If you are not a chess player, you will be unable to decipher the meaning of the moves. But once you learn the rules of the game and the meaning of the symbols (R = rook, B = bishop, Q = queen, . . ., x = "capture", etc.) and the convention of naming the squares of the chessboard, you will be able to reproduce the moves of the game as it was played. Not only that. Most likely, you will be able to follow the analysis of the game after move 24, for example. Such an analysis can be thought of as the chess equivalent of a proof of a "lemma" or "theorem" in mathematics. For example, the question mark affixed to black's move 24 is a claim that it is a bad move, and the following commentary is the analyst's explanation why it is a bad move. Almost always, it is possible to read the explanation and understand it completely. There is also no need to turn to some secondary thought process involving perhaps the analyses of other games or certain middlegame theory. Of course, the ability to understand the analysis of the game does not imply the ability to play chess well.

In mathematics, the situation is more daunting. Having studied a certain field for a few years, you may not even be able to read and understand a mathematical paper unless you are working on a similar problem. Often this is also the case for specialists in a related field. Not to mention the layman.

A language for all seasons...

True, every discipline has its own written language - economics, sociology, physics, chemistry, biology to name a few. By comparison, the characteristics of the mathematical language are its cohesiveness, coherence and closed nature. Other fields of knowledge are often clothed in a language that is either mathematical or else a dialect of mathematics. It is no exaggeration to say that the language of physics is mathematics. When quantifiable concepts are introduced into a discipline, mathematical terms invariably creep in. Even in sociology, concepts of linear algebra have been used. So pervasive and effective is the use of mathematical concepts in physics that the Nobel laureate Eugene Paul Wigner (1902 – 1995) was prompted to write:

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor

deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure even though perhaps also to our bafflement, to wide branches of learning.

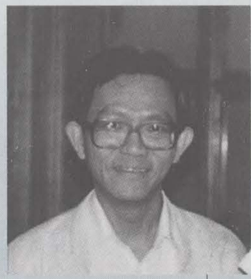
The language of mathematics has also found its way into our daily lives. As society becomes more developed and highly structured and as more people become more affluent with more leisure at their disposal, the need to understand and use mathematical terminology in daily life becomes more imperative. At first, you will be thinking and talking about "averages", "chances", "expectations", "optimal strategy", "tautology", "consistency", for example, in qualitative terms. But soon, you may be using them in more precise (and almost mathematical) terms.

The inexorable intrusion of computers into public (and some private) aspects of life has made the need to come to terms with a hybrid dialect of mathematics even more pressing. It is possible to minimize contact with mathematics after having fulfilled one's obligations at school or university – with one's significant contact occurring during the annual income tax assessment exercise. Yet it is also possible to maintain a life-long contact with mathematics in a meaningful and non-threatened way.

Short of a return to a dark age of irrationality, there seems to be no turning back from the crossroads that lead to a mathematicizing of daily life. The infusion of mathematical ideas into our thinking can only increase clarity of thought and lead to a general rise in rationality of action. It is not so much the coldness of logic or the indifference of computing that should worry us. It is the failure to absorb the richness in imagination or the diversity in conceptualization into our mental states that should give us concern.

For a few, mathematics is a form of poetry. For some, mathematics provides a different eye-piece with which to view life from a multi-dimensional perspective. But for many of us, the language of mathematics is no different from that of Egyptian hieroglyphics - just as distant and esoteric and existing in another world. Even for those who have learnt to decipher its meaning during a few intense years, not many are able to penetrate beneath the writing on the board to feel the vibrancy and dynamism of the language. Mastery of its syntax seems to take a long time, and without this mastery, the mastery of applications seems just as remote.

For many people mathematics is usually associated with the solving of problems which seem to be contrived and removed from reality. Perhaps, if we look at mathematics as a language with which we can use to view and describe life and nature in hues and shades otherwise unavailable, we will be able to open our minds to see the gentler side of mathematics. M²



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