In our preceding article [1], we introduced the celebrated Ceva's Theorem and its converse which is stated as follows:

The cevians $AP$, $BQ$ and $CR$ of $	riangle ABC$ are concurrent if and only if

$$\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1.$$  

Figure 1

Three distinct points on a plane are said to be collinear if they lie on a straight line. Given $	riangle ABC$, let $X$, $Y$ and $Z$ be, respectively, points other than the vertices $A$, $B$, $C$, on the lines formed from sides $BC$, $CA$ and $AB$ as shown in Figure 2. Ceva's theorem and its converse provide us with a criterion to determine whether three given cevians are concurrent. We may ask: is there a criterion which will enable us to determine whether the three given points as shown in Figure 2 are collinear?

While Ceva's theorem was established in the 17th century, a positive answer to the above question was given two thousand years ago by Menelaus of Alexandria (about 98A.D.). In this article, we shall introduce this important result and also show some of its applications.

### Menelaus' Theorem.

Let $ABC$ be a triangle, and let $X$, $Y$ and $Z$ be points on the lines formed from $BC$, $CA$ and $AB$ respectively as shown in Figure 2. If $X$, $Y$ and $Z$ are collinear, then

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1.$$  

(1)
There are several different proofs of Menelaus' theorem. In what follows, we
give two of them; the first proof applies the notion of area, and the second proof
uses the ratio theorem.

**First Proof**
We denote by \((PQR)\) the area of \(\triangle PQR\).

Consider Figure 3. As was shown in [1], we have

\[
\frac{AZ}{(AYZ)} = \frac{ZB}{(BYZ)} \quad \frac{BX}{(BYZ)} = \frac{XC}{(CYZ)}
\]

and

\[
\frac{CY}{(CYZ)} = \frac{YA}{(AYZ)} .
\]

Thus

\[
\frac{AZ}{(AYZ)} \cdot \frac{BX}{(BYZ)} \cdot \frac{CY}{(CYZ)} = \frac{(AYZ)}{(BYZ)} \cdot \frac{(BYZ)}{(CYZ)} \cdot \frac{(CYZ)}{(AYZ)} = 1,
\]

as required.

**Second Proof**
As shown in Figure 4, let \(D\) be the point on the line formed from \(CA\) such that

\(BD//XY\). Then by the ratio theorem, we have:

\[
\frac{AZ}{AY} = \frac{ZB}{YD} \quad \frac{BX}{DY} = \frac{XC}{YC}
\]

Thus

\[
\frac{AZ}{AY} \cdot \frac{BX}{DY} \cdot \frac{CY}{YC} = \frac{(AYZ)}{(BYZ)} \cdot \frac{(BYZ)}{(CYZ)} \cdot \frac{(CYZ)}{(AYZ)} = 1,
\]

as desired. \(\square\)

We shall now give two examples to illustrate the use of Menelaus’ theorem.

**Example 1**
In Figure 5, \(ABC\) is a triangle with \(\angle B = 90^\circ\), \(BC = 3\text{cm}\) and \(AB = 4\text{cm}\). \(D\) is
a point on \(AC\) such that \(AD = 1\text{cm}\), and \(E\) is the mid-point of \(AB\). Join \(D\) and \(E\), and extend \(DE\) to meet \(CB\) extended at \(F\). Find \(BF\).

**Solution**
Consider \(\triangle ABC\). Then \(D, E\) and \(F\) are, respectively, points on the sides \(CA, AB\)
and \(BC\), and by construction are collinear. By Menelaus’ theorem,

\[
\frac{AE}{EB} \cdot \frac{BF}{FC} \cdot \frac{CD}{DA} = 1. \quad (i)
\]

By assumption, \(AE = EB = 2\), \(DA = 1\) and \(FC = FB + BC = BF + 3\). By
Pitogoras’ theorem,

\[
AC = \sqrt{BC^2 + AB^2} = \sqrt{3^2 + 4^2} = 5,
\]

and so \(CD = AC - AD = 5 - 1 = 4\). Substituting these data into (i) gives

\[
\frac{2}{BF} \cdot \frac{4}{BF + 3} = 1.
\]

Solving for \(BF\) yields \(BF = 1\). \(\square\)
In applying Menelaus' theorem, we need to identify a triangle and three collinear points respectively on its sides. (Thus, in Example 1, we take ΔABC and the points D, E and F.) To simplify notation, in what follows, in Menelaus' theorem we refer to the lines YZX in Figure 2(a) and ZXY in Figure 2(b) as the transversals of ΔABC.

Example 2

In Figure 6, ABC is a triangle, X and Y are points on BC and CA respectively, and R is the point of intersection of AX and BY.

Given \( \frac{AY}{YC} = p \) and \( \frac{AR}{RX} = q \), where \( 0 < p < q \), express \( \frac{BX}{XC} \) in terms of \( p \) and \( q \).

Solution

Consider ΔAXC and its transversal BRY. By Menelaus' theorem,

\[
\frac{AR}{RX} \cdot \frac{XB}{BC} \cdot \frac{CY}{YA} = 1.
\]

Thus

\[
\frac{BC}{XB} = \frac{AR}{RX} \cdot \frac{CY}{YA} = q
\]

i.e.,

\[
\frac{BX + XC}{BX} = q.
\]

It follows that

\[
1 + \frac{XC}{BX} = q,
\]

\[
\frac{XC}{BX} = q - p
\]

i.e.,

\[
\frac{BX}{XC} = \frac{p}{q - p}.
\]

Let X, Y and Z be, respectively, points on the sides BC, CA and AB of ΔABC as shown in Figure 2. Menelaus' theorem states that if X, Y and Z are collinear, then equality (1) holds. Does the converse of Menelaus' theorem also hold? That is, if X, Y and Z are points such that equality (1) holds, are they always collinear? A positive answer to this question is given in the following result.

The Converse of Menelaus' Theorem.

Let X, Y and Z be points on the lines formed from the sides BC, CA and AB of ΔABC respectively.

If

\[
\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = 1,
\]

then X, Y and Z are collinear.

The proof of the above result is similar to the proof of the converse of Ceva's theorem as given in [1]. We leave the proof of the above result to the reader.

The converse of Menelaus' theorem is very useful in showing the collinearity of three given points on a plane. Two examples are given below.
Example 3

In Figure 7, the diagonals $AC$ and $BD$ of a quadrilateral $ABCD$ meet at $M$ in such a way that $AM = MC$ and $DM = 2MB$. Suppose that $X$ and $Y$ are points on $MC$ and $BC$ respectively such that

$$\frac{AC}{MX} = \frac{BY}{YC} = 3.$$

Show that the points $D$, $X$ and $Y$ are collinear.

Proof

First, we have

$$\frac{DM}{BD} = \frac{DM}{BM + MD} = \frac{2MB}{3MB} = \frac{2}{3},$$

i.e., $\frac{DM}{BD} = \frac{2}{3}$. (i)

Next,

$$\frac{CX}{XM} = \frac{CM - XM}{XM} = 1 + \frac{AC}{XM} = 3,$$

i.e., $\frac{CX}{XM} = \frac{1}{2}$. (ii)

Now, consider $\triangle MBC$ and the points $D$, $X$ and $Y$. By (i), (ii) and using the assumption $\frac{BY}{YC} = 3$, $\frac{BY}{YD} = 3$, $\frac{CX}{XM} = \frac{1}{2}$, and $\frac{MD}{DB} = \frac{1}{2}$, we have

$$\frac{BY}{YC} = 3,\quad \frac{CX}{XM} = \frac{1}{2},\quad \frac{MD}{DB} = \frac{1}{2}.$$

Hence, by the converse of Menelaus' theorem, $D$, $X$ and $Y$ are collinear.}$

Girard Desargues (1591–1661), a French architect, discovered an important and interesting result relating the collinearity of points and concurrency of lines on two triangles, which became a fundamental result in Projective Geometry. We shall now state this result and prove it by applying both Menelaus' theorem and its converse.

### Desargues' Theorem.

Let $ABC$ and $A'B'C'$ be two given triangles such that the lines $AA'$, $BB'$ and $CC'$ are concurrent, as shown in Figure 8. Let $X$, $Y$ and $Z$ be, respectively, the points of intersection of the lines $AB$ and $A'B'$, $BC$ and $B'C'$ and $CA$ and $C'A'$. Then $X$, $Y$ and $Z$ are collinear.
Proof

Observe that $X$, $Y$ and $Z$ are points on the lines formed from the sides $AB$, $BC$ and $CA$ of $\triangle ABC$ respectively. Thus, to show that $X$, $Y$ and $Z$ are collinear, by the converse of Menelaus’ theorem, it is enough to show that

$$\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = 1.$$

First, consider $\triangle ANB$ and its transversal $A'B'X$. By Menelaus’ theorem,

$$\frac{NA'}{AX} \cdot \frac{X'B'}{XB} \cdot \frac{BY}{YN} = 1. \quad (i)$$

Next, consider $\triangle NBC$ and its transversal $Y'B'C'$. By Menelaus’ theorem,

$$\frac{NB'}{BY} \cdot \frac{Y'C'}{YC} \cdot \frac{C'A'}{CA} = 1. \quad (ii)$$

Now, consider $\triangle NCA$ and its transversal $Z'A'C'$. By Menelaus’ theorem,

$$\frac{NC'}{CZ} \cdot \frac{ZA'}{AA} \cdot \frac{AY}{YN} = 1. \quad (iii)$$

Finally, the product of (i), (ii) and (iii) gives

$$\frac{AX}{XB} \cdot \frac{BY}{YC} \cdot \frac{CZ}{ZA} = 1,$$

as was to be shown. \[Q.E.D.\]

We end this article by giving the following final example, which is actually Question 3 of the 1989 Asian Pacific Mathematics Olympiad. (Ten students from Singapore took part in this competition. Seven of them, Lam Vui Chiap, Lee Mun Yew, Loh Ngai Seng, Ng Lup Keen, Yan Weide, Yeo Don and Yeoh Yong Yeow, managed to solve this question completely. The common feature of their solutions was the use of Menelaus’ theorem. We present here an outline of one of these approaches. The reader is invited to fill in any gaps.)

Example 4

Let $A_1$, $A_2$, $A_3$ be three points in the plane, and for convenience, let $A_4 = A_1$, and $A_5 = A_2$. For $n = 1, 2$ and 3, suppose that $B_n$ is the midpoint of $A_nA_{n+1}$ and suppose that $C_n$ is the midpoint of $A_nB_n$. Suppose that $A_1A_2$, and $B_1B_2$ meet at $D_1$ and that $A_3A_4$, and $C_3C_4$ meet at $E_1$. Calculate the ratio of the area of triangle $O_1O_2O_3$ to the area of triangle $E_1E_2E_3$.

Solution

Our aim is to compute the values of $\frac{(D_1D_2D_3)}{(A_1A_2A_3)}$ and $\frac{(E_1E_2E_3)}{(A_1A_2A_3)}$, from which we can immediately determine the value of $\frac{(D_1D_2D_3)}{(E_1E_2E_3)}$.

Consider $\triangle A_1A_2B_1$, and its transversal $A_1D_1C_1$ (see Figure 9). By Menelaus’ theorem,

$$\frac{A_1C_1}{C_1A_1} \cdot \frac{A_2D_1}{D_1A_2} \cdot \frac{B_1A_1}{A_1A_2} = 1. \quad (i)$$

As $A_2C_1 = \frac{1}{3} A_1D_1$, and $B_1A_1 = \frac{1}{2} A_1A_2$, it follows from (i) that

$$B_1D_1 = \frac{1}{6} A_1D_1,$$

and so $B_1D_1 = \frac{1}{7} A_1B_1$. \[ii\]

Let $G$ denote the centroid of $\triangle A_1A_2A_3$; then

$$G B_1 = \frac{1}{3} A_1B_1.$$ \[iii\]

Thus $GD_1 = GB_1 - B_1D_1$.
\[
\begin{align*}
&= \left( \frac{1}{3} \cdot \frac{1}{2} \right) A_i B_i \quad \text{(by (ii) and (iii))} \\
&= \frac{4}{21} A_i B_i \\
&= \frac{4}{21} \cdot \frac{3}{2} G A_i \quad \text{(by (iii))} \\
&= \frac{2}{7} G A_i, \\
\end{align*}
\]

i.e., \( GD_1 = \frac{2}{7} G A_i \). \quad \text{(iv)}

Likewise, \( GD_2 = \frac{2}{7} G A_i \) \quad \text{(v)}

and \( GD_3 = \frac{2}{7} G A_i \). \quad \text{(vi)}

It follows from (iv) and (v) that

\[ \Delta G D_1 D_2 - \Delta G A D_1 A, \]

and so

\[ \left( \frac{G D_1 D_2}{G A A_1} \right)^2 = \left( \frac{2}{7} \right)^2 = \frac{4}{49}. \quad \text{(vii)} \]

Likewise,

\[ \left( \frac{G D_1 D_2}{G A A_1} \right)^2 = \left( \frac{G D_1 D_2}{G A A_1} \right)^2 = \frac{4}{49}. \quad \text{(viii)} \]

Combining (vii) and (viii) yields

\[ \left( \frac{D_1 D_2 D_3}{A_1 A_2 A_3} \right)^2 = \frac{4}{49}. \quad \text{(ix)} \]

Next, consider \( \Delta A_1 A_2 B_2 \) and its transversal \( A_1 C_1 A_2 \). By Menelaus' theorem,

\[ \frac{A_1 C_1}{C_1 A_2} \cdot \frac{A_1 A_2}{A_2 B_2} \cdot \frac{B_2 E}{E A_1} = 1. \]

As \( \frac{A_1 C_1}{C_1 A_2} = \frac{1}{3} \) and \( \frac{A_1 A_2}{A_2 B_2} = 2, \)

we have \( A_1 E_1 = \frac{2}{3} B_2 E_1, \)

and so \( A_1 E_1 = \frac{2}{5} A_1 B_1 = \frac{2}{5} \cdot \frac{3}{2} G A_i = \frac{3}{5} G A_i. \)

Thus \( G E_1 = G A_i - A_1 E_1 = G A_i - \frac{3}{5} G A_i = \frac{2}{5} G A_i. \)

Similarly, \( G E_2 = \frac{2}{5} G A_i \) and \( G E_3 = \frac{2}{5} G A_i. \)

Following a similar argument as given in the first part, we have

\[ \left( \frac{E_1 E_2 E_3}{A_1 A_2 A_3} \right)^2 = \frac{4}{25}. \quad \text{(x)} \]

Combining (ix) and (x) yields

\[ \left( \frac{D_1 D_2 D_3}{E_1 E_2 E_3} \right)^2 = \frac{25}{49}. \]

\[ \]

Reference