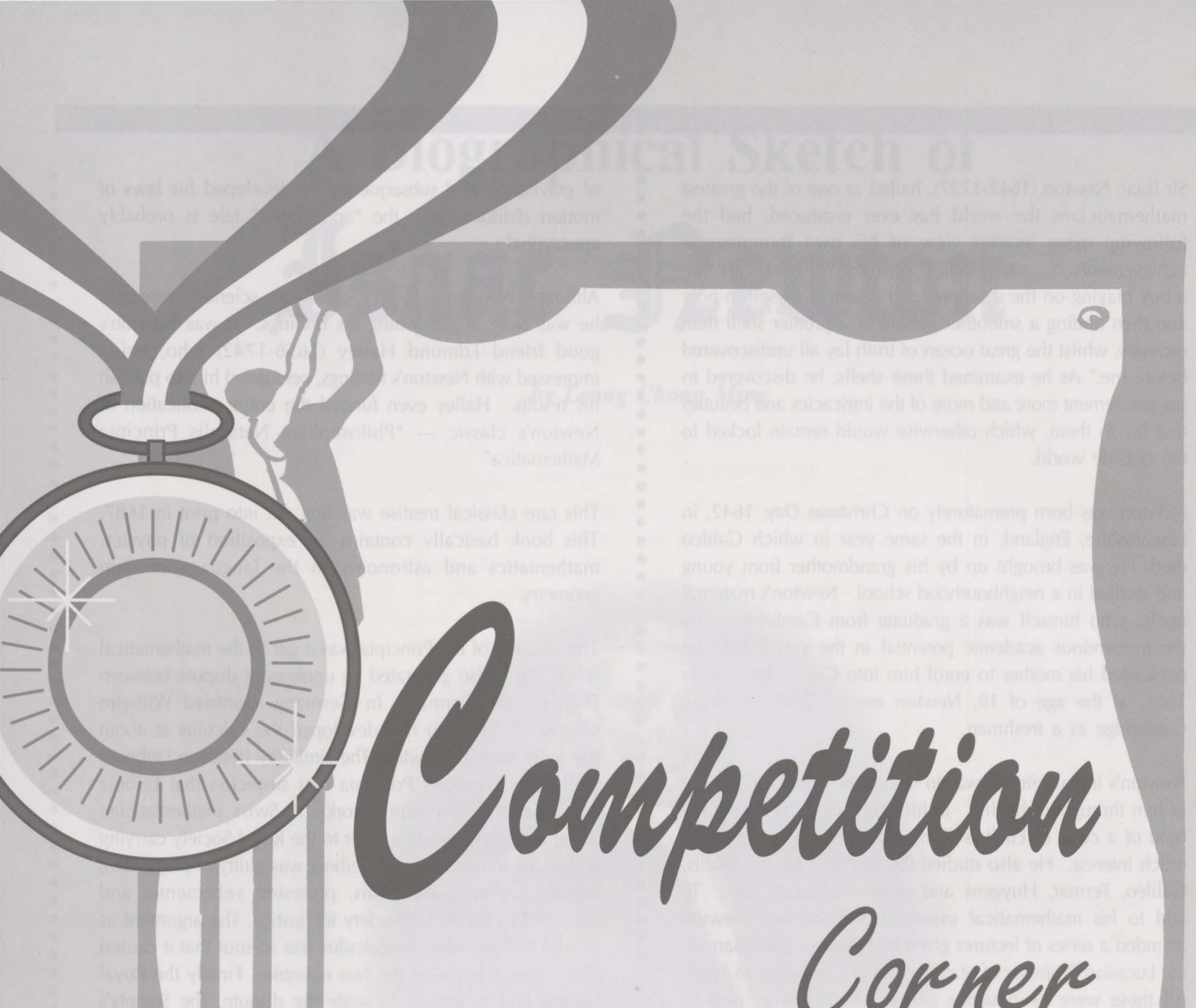

Competition Corner

by Tay Tiong Seng

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In this issue we publish the problems of the second round of the 13th Iranian National Mathematics Olympiad. These problems were sent to us by Mohamad Mahdian who is a member of the Iranian National Olympiad Committee. He did not mention the time allowed but I guess it is along the line of the International Mathematical Olympiad, that is the contest was held over two days with 3 problems each day and the time allowed was 4.5 hours. There were 187 participants in the second round of competition. These were chosen from the 13,000 participants who took part in the first round. The second round was held in Yazd, a beautiful ancient city of Iran. In the second round, 11 gold medals, about 20 silver and about 30 bronze medals were awarded. Two of the gold medalists were less than 15 years old. The six contestants represented Iran in the International Mathematical Olympiad 1996 held in India were chosen from the gold medalists. Incidentally, in this Olympiad Iran obtained 1 gold, 4 silver and 1 bronze medals and was ranked eighth unofficially. I am also including solutions to two of the more difficult problems. Readers are urged to send in their solutions to the problems. All contributors with correct solutions will be acknowledged.

Problems of the 13th Iranian National Mathematics Olympiad (INMO'95)

Second Round

1. Show that for every $n > 3$ there exist two sets of integers $A = \{x_1, x_2, ..., x_n\}$, $B = \{y_1, y_2, ..., y_n\}$, such that:
   (i) A and B are disjoint.
   (ii) $x_1 + x_2 + ... + x_n = y_1 + y_2 + ... + y_n$.
   (iii) $x_1^2 + x_2^2 + ... + x_n^2 = y_1^2 + y_2^2 + ... + y_n^2$.

2. ABC is an acute angled triangle and L is a line in its plane. The line L is reflected about each of the sides of ABC. The three resulting lines intersect pairwise to form a triangle $A'B'C'$. Prove that the incenter of $\Delta A'B'C'$ lies on the circumcircle of ΔABC.

3. In a party there are $12k$ guests. Every guest knows exactly $3k + 6$ other guests. Suppose that if x knows y, then y knows x too. For every two guests x and y in this party there are exactly n guests who know both x and y. (n is a constant)
   Determine the number of guests in this party.

4. Let $S = \{2^m 3^n \mid m, n$ are nonnegative integers$\}$. Prove that every natural number can be written as a sum of distinct elements of S such that none of them is a multiple of another.

5. Prove that for every positive integer n
   $$[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}] = [\sqrt{9n+8}]$$
   where $[x]$ denotes the smallest integer greater than or equal to x.

6. In a tetrahedron $ABCD$, let A', B', C', and D' be the circumcentres of ΔBCD, ΔCDA, ΔDAB, and ΔABC, respectively. Denote the plane which passes through the point X and is perpendicular to the line YZ by $S(X, YZ)$. Prove that the four planes $S(A, C'D')$, $S(B, D'A')$, $S(C, A'B')$ and $S(D, B'C')$, have a point in common.

Solutions to Problems 3 and 6

3. Form the graph where the vertices are the guests and where two vertices are adjacent if and only if the guests concerned know each other. Then there are $v = 12k$ vertices and $\deg(x) = 3k + 6$ for every vertex x. (Note $\deg(x)$ is the number of edges adjacent with x.) For each pair of vertices x, y let $\deg(x, y)$ denote the number of vertices that are adjacent to both x and y. Thus $\deg(x, y) = n$. Since there are $\binom{12k}{2}$ pairs of vertices, we have
   $$\sum \deg(x, y) = n\binom{12k}{2}$$
   where the summation is over all pairs $\{x, y\}$ of vertices.

   For each vertex x, since $\deg(x) = 3k + 6$ there are $\binom{3k+6}{2}$ pairs of vertices which are adjacent to x. Thus
   $$\sum \deg(x, y) = 12k\binom{3k+6}{2}$$

Consequently we have

$$9k^2 + (33 - 12n)k + (30 + n) = 0.$$

This implies that 6 divides n. Treating the above as a quadratic equation in k, since the solution is an integer, the discriminant is a perfect square.

Thus for some integer p, we have

$$16n^2 - 92n = p^2 - 1.$$

By completing squares we have

$$(8n - 23)^2 - (2p)^2 = 525.$$

The left hand side is $(8n - 23 - 2p)(8n - 23 + 2p)$. Thus in the corresponding factorization of the right hand side into two factors $a \times b$, we have $a + b = 16n - 46$. Since n is a multiple of 6, we have $a + b \equiv 2 \pmod 3$. There are six factorizations of 525 into two factors of which only 5×105 and 15×35 have the desired property. The first gives $n = 39$ while the second gives $n = 6$. Consequently, $k = 3$ and there are 36 guests in the party.

6. (*Solution due to Jeffrey Pang Chin How, Hwa Chong Junior College. Jeffrey also represented Singapore in the International Mathematical Olympiad in 1995 and 1996 and won a Bronze medal each time.*)

We first state the following result which is easy to prove: Let P, Q, R, S be four coplanar points. Then PQ is perpendicular to RS if and only if $PR^2 - PS^2 = RQ^2 - QS^2$.

Let r_A be the circumradius of triangle BCD. The quantities r_B, r_c and r_D are similarly defined.

Let Y be the point of intersection of $C'D'$ and $S(A, C'D')$. Then a point X is on $S(A, C'D')$ if and only if XY is perpendicular to $C'D'$. Thus X is on the plane $S(A, C'D')$ if and only if

$$\begin{aligned} XC'^2 - XD'^2 &= C'Y^2 - D'Y^2 \\ &= C'A^2 - D'A^2 \\ &= r_C^2 - r_D^2 . \end{aligned}$$

It is known that three planes in space have a point in common. Let P be a point which is on $S(A, C'D')$, $S(B, D'A')$ and $S(C, A'B')$. Then

$$PC'^2 - PD'^2 = r_C^2 - r_D^2,$$
$$PD'^2 - PA'^2 = r_D^2 - r_A^2,$$
$$PA'^2 - PB'^2 = r_A^2 - r_B^2 .$$

Thus we have

$$PC'^2 - PB'^2 = r_C^2 - r_B^2$$

Hence P is on $S(D, B'C')$ M²

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