Graphics Calculator and the Inequality

$$p^{r+k} + q^{r+k} \geq p^r q^k + q^r p^k$$

Ho Foo Him and Spario Soon

Curve sketching is an important component in the GCE "A" Level mathematics syllabi. One of the strategies for helping students to understand this topic well is to motivate them to develop a special interest in this area. We feel strongly that using computing technology such as a graphics calculator or a computer software helps students in understanding and exploring different aspects of curve sketching.

The first author taught a group of students using the TI82 graphics calculator to investigate the curves $y = x^k$, with special attention given to positive values of x. By changing the value of k, where k is a positive real number, the students could actually observe some characteristics of the curves. For example, some of the characteristics they wrote down for $y = x^k$, where x is positive, are

1. the curve is increasing when $x \geq 0$,
2. when $k < 1$, the curves are concave upward, and
3. when $0 < k < 1$, the curves are concave downward

The curves $y = x^2$ and $y = x^{\frac{1}{2}}$ were sketched on the same diagram (see figure 1) and investigated by using the "TABLE" function of the calculator (see figure 2). Some students observed that for any two distinct positive numbers p *and* q, $p^2 - q^2$ and $p^{\frac{1}{2}} - q^{\frac{1}{2}}$ have the same sign.

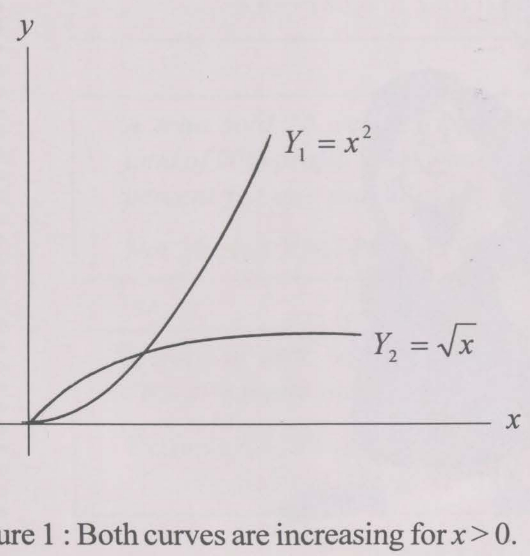

Figure 1 : Both curves are increasing for $x > 0$.

X	Y_1	$Y2$
.5	.25	.70711
.6	.36	.7746
.7	.49	.83666
.8	.64	.89443
.9	.81	.94868
1	1	1
1.1	1.21	1.0488
$X = .8$		

Figure 2 : The "TABLE" function allows the student to claim that the signs of $p^2 - q^2$ and $p^{\frac{1}{2}} - q^{\frac{1}{2}}$ are the same.

This remark prompted them to think further. With a little guidance and explorations, students went on to generalise that for any positive real numbers p, q, r and k, the signs of $p^r - q^r$ and $p^k - q^k$ are the same.

This further implies that $(p^r - q^r)(p^k - q^k) \geq 0$.

That is: $$p^{r+k} - p^r q^k - q^r p^k + q^{r+k} \geq 0$$

or $$p^{r+k} + q^{r+k} \geq p^r q^k + q^r p^k \qquad (1)$$

where equality holds when $p = q$.

We have obtained an inequality from a seemingly unrelated lesson of curve sketching! Furthermore, as the inequality is constructed by the students, they can understand and remember the inequality well. In fact they are motivated to "creating" many more related inequalities based on (1).

The following are some inequalities that were obtained based on (1).
By letting $r = 1$ and $k = 2$ in (1), we have $p^3 + q^3 \geq pq(p + q)$And by letting $r = 1$and $k = \frac{1}{2}$, we have

$$p^{\frac{3}{2}} + q^{\frac{3}{2}} \geq p^{\frac{1}{2}} q + q^{\frac{1}{2}} p. \qquad (2)$$

Dividing $\sqrt{pq}$ throughout in (2), it becomes $\frac{p}{\sqrt{q}}+\frac{q}{\sqrt{p}} \ge \sqrt{p}+\sqrt{q}$. Also, when dividing (2) throughout by pq, we have $\frac{\sqrt{q}}{p}+\frac{\sqrt{p}}{q} \ge \frac{1}{\sqrt{p}}+\frac{1}{\sqrt{q}}$.

Let us now try to generalsie inequality (1) by investigating $\left(\sum_{i=1}^{n} x_i^r\right)\left(\sum_{j=1}^{n} x_j^k\right)$.

As $\left(\sum_{i=1}^{n} x_i^r\right)\left(\sum_{j=1}^{n} x_j^k\right) = \sum_{1\le i<j\le n}\left(x_i^r x_j^k + x_i^k x_j^r\right) + \sum_{i=1}^{n} x_i^{r+k}$ and

$$\sum_{1\le i<j\le n}\left(x_i^r x_j^k + x_i^k x_j^r\right) \le \sum_{1\le i<j\le n}\left(x_i^{r+k} + x_j^{r+k}\right) \qquad \text{(by (1))}$$

we have

$$\left(\sum_{i=1}^{n} x_i^r\right)\left(\sum_{j=1}^{n} x_j^k\right) \le \sum_{1\le i<j\le n}\left(x_i^{r+k} + x_j^{r+k}\right) + \sum_{i=1}^{n} x_i^{r+k}$$

$$\le (n-1)\sum_{i=1}^{n} x_i^{r+k} + \sum_{i=1}^{n} x_i^{r+k} = n\sum_{i=1}^{n} x_i^{r+k}.$$

Hence, we have obtained the inequality

$$x_1^{r+k} + x_2^{r+k} + x_3^{r+k} + \cdots + x_n^{r+k} \ge \frac{1}{n}\left(x_1^r + x_2^r + x_3^r + \cdots + x_n^r\right)\left(x_1^k + x_2^k + x_3^k + \cdots + x_n^k\right), \qquad (3)$$

where x_i, r and k are positive real numbers. Note that when $n = 2$, the above inequality becomes (1)

Conclusion

To observe, hypothesize and finally, to justify, is an important process in the study of mathematics. In the above classroom example, a routine lession of plotting graphs of functions was made fun and easy with the help of a graphics calculator. And by observing the behaviour of graphs of a class of functions, the students were led to the "discovery" of some interesting inequalities. We believe that a tool such as a graphics calculator, when used appropriately in the teaching of mathematics, can help to inspire our students to think, to make correlations and to explore and discover new results in the process of learning mathematics both inside and outside the classroom. We encourage all teachers to incorporate computing technology in the classroom.

Editor's Remark

Inequality (3) is a special form of Chebyshev's inequality: suppose

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \text{ and } b_1 \geq b_2 \geq \cdots \geq b_n \geq 0.$$

Then

$$\sum_{i=1}^{n} a_i b_i \geq \frac{1}{n}\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} b_j\right).$$

To deduce (3), take $x_i = a_i = b_i$, and observe that the order of x_i's are not important in this case. In fact, Chebyshev's inequality can be proved in this way. Simply observe that as $(a_i - a_j)(b_i - b_j) \geq 0$, we have $a_i b_i + a_j b_j \geq a_i b_j + b_i a_j$. The rest goes through as in the proof of (3).

Mr Ho Foo Him has many years of experience in teaching mathematics at junior colleges and is currently the Head of Information Technology of Serangoon Junior College. He was a member of the Singapore Mathematical Olympiad Training Committee for several years and was the deputy leader of the Singapore International Mathematical Olympiad team in 1991. He has a keen interest in exploring how technology can be used in Mathematics teaching and learning.

Dr Spario Y. T. Soon has taught in schools and institutions for years. He was a lecturer for Mathematics at the National Institute of Education. In 1997, he received the Nanyang Technology University's Teacher of the Year award. Currently, he is a mathematics consultant at the Interactive Math Exploration Centre.