Graphics Calculator and the Inequality

\[ p^r + k + q^r + k \geq p^r q^k + q^r p^k \]

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Curve sketching is an important component in the GCE “A” Level mathematics syllabi. One of the strategies for helping students to understand this topic well is to motivate them to develop a special interest in this area. We feel strongly that using computing technology such as a graphics calculator or a computer software helps students in understanding and exploring different aspects of curve sketching.

The first author taught a group of students using the TI82 graphics calculator to investigate the curves \( y = x^k \), with special attention given to positive values of \( x \). By changing the value of \( k \), where \( k \) is a positive real number, the students could actually observe some characteristics of the curves. For example, some of the characteristics they wrote down for \( y = x^k \), where \( x \) is positive, are

1. the curve is increasing when \( x \geq 0 \),
2. when \( k < 1 \), the curves are concave upward, and
3. when \( 0 < k < 1 \), the curves are concave downward.

The curves \( y = x^2 \) and \( y = x^\frac{1}{2} \) were sketched on the same diagram (see figure 1) and investigated by using the “TABLE” function of the calculator (see figure 2). Some students observed that for any two distinct positive numbers \( p \) and \( q \), \( p^2 - q^2 \) and \( \frac{p}{2} - \frac{q}{2} \) have the same sign.

\[
\begin{array}{|c|c|c|}
\hline
X & Y_1 & Y_2 \\
\hline
.5 & .25 & .70711 \\
.6 & .36 & .7746 \\
.7 & .49 & .83666 \\
.8 & .64 & .89443 \\
.9 & .81 & .94868 \\
1 & 1 & 1 \\
1.1 & 1.21 & 1.0488 \\
\hline
\end{array}
\]

Figure 2: The “TABLE” function allows the student to claim that the signs of \( p^2 - q^2 \) and \( \frac{p}{2} - \frac{q}{2} \) are the same.

This remark prompted them to think further. With a little guidance and explorations, students went on to generalise that for any positive real numbers \( p, q, r \) and \( k \), the signs of \( p^r - q^r \) and \( p^k - q^k \) are the same.

This further implies that \( (p^r - q^r)(p^k - q^k) \geq 0 \).

That is:

\[
p^{r+k} - p^r q^k - q^r p^k + q^{r+k} \geq 0
\]

or

\[
p^{r+k} + q^{r+k} \geq p^r q^k + q^r p^k \quad (1)
\]

where equality holds when \( p = q \).

We have obtained an inequality from a seemingly unrelated lesson of curve sketching! Furthermore, as the inequality is constructed by the students, they can understand and remember the inequality well. In fact they are motivated to “creating” many more related inequalities based on (1).

The following are some inequalities that were obtained based on (1).

By letting \( r = 1 \) and \( k = 2 \) in (1), we have

\[
p^3 + q^3 \geq pq(p + q)
\]

And by letting \( r = 1 \) and \( k = \frac{1}{2} \), we have

\[
p^\frac{3}{2} + q^\frac{3}{2} \geq p^\frac{1}{2} q + q^\frac{1}{2} p
\]

Figure 1: Both curves are increasing for \( x > 0 \).
Dividing $\sqrt{pq}$ throughout in (2), it becomes $\frac{p}{\sqrt{q}} + \frac{q}{\sqrt{p}} \geq \sqrt{p} + \sqrt{q}$. Also, when dividing (2) throughout by $pq$, we have $\frac{\sqrt{q}}{p} + \frac{\sqrt{p}}{q} \geq \frac{1}{\sqrt{p}} + \frac{1}{\sqrt{q}}$.

Let us now try to generalize inequality (1) by investigating \( \left( \sum_{i=1}^{n} x_i^r \right) \left( \sum_{j=1}^{n} x_j^i \right) \).

As \( \left( \sum_{i=1}^{n} x_i^r \right) \left( \sum_{j=1}^{n} x_j^i \right) = \sum_{1 \leq i < j \leq n} (x_i^r x_j^i + x_j^r x_i^j) + \sum_{i=1}^{n} x_i^{r+k} \) and

\[
\sum_{1 \leq i < j \leq n} (x_i^r x_j^i + x_j^r x_i^j) \leq \sum_{1 \leq i < j \leq n} (x_i^{r+k} + x_j^{r+k}). \quad \text{(by (1))}
\]

we have

\[
\left( \sum_{i=1}^{n} x_i^r \right) \left( \sum_{j=1}^{n} x_j^i \right) \leq \sum_{1 \leq i < j \leq n} (x_i^{r+k} + x_j^{r+k}) + \sum_{i=1}^{n} x_i^{r+k}
\]

\[
\leq (n-1) \sum_{i=1}^{n} x_i^{r+k} + \sum_{i=1}^{n} x_i^{r+k} = n \sum_{i=1}^{n} x_i^{r+k}.
\]

Hence, we have obtained the inequality

\[
x_1^{r+k} + x_2^{r+k} + x_3^{r+k} + \ldots + x_n^{r+k} \geq \frac{1}{n} \left( x_1^r + x_2^r + x_3^r + \ldots + x_n^r \right) \left( x_1^i + x_2^i + x_3^i + \ldots + x_n^i \right), \quad \text{(3)}
\]

where $x_i, r$ and $k$ are positive real numbers. Note that when $n = 2$, the above inequality becomes (1)

**Conclusion**

To observe, hypothesize and finally, to justify, is an important process in the study of mathematics. In the above classroom example, a routine lesson of plotting graphs of functions was made fun and easy with the help of a graphics calculator. And by observing the behaviour of graphs of a class of functions, the students were led to the “discovery” of some interesting inequalities. We believe that a tool such as a graphics calculator, when used appropriately in the teaching of mathematics, can help to inspire our students to think, to make correlations and to explore and discover new results in the process of learning mathematics both inside and outside the classroom. We encourage all teachers to incorporate computing technology in the classroom.
Editor’s Remark

Inequality (3) is a special form of Chebyshev’s inequality: suppose

\[ a_1 \geq a_2 \geq \cdots \geq a_n \geq 0 \text{ and } b_1 \geq b_2 \geq \cdots \geq b_n \geq 0. \]

Then

\[ \sum_{i=1}^{n} a_i b_i \geq \frac{1}{n} \left( \sum_{j=1}^{n} a_j \right) \left( \sum_{j=1}^{n} b_j \right). \]

To deduce (3), take \( x_i = a_i = b_i \), and observe that the order of \( x_i \)'s are not important in this case. In fact, Chebyshev’s inequality can be proved in this way. Simply observe that as \( (a_i - a_j)(b_i - b_j) \geq 0 \), we have \( a_i b_i + a_j b_j \geq a_i b_j + a_j b_i \). The rest goes through as in the proof of (3).

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