Some Applications of the Discriminant of a Quadratic Function

by Ho Foo Him
Given a quadratic function $f(x) = ax^2 + bx + c$, where $a \neq 0$, $a$, $b$, and $c$ are real numbers, its discriminant, $D$, is defined as $b^2 - 4ac$. In this note, we will look at some nice and interesting applications of the discriminant which are normally not included in a secondary school mathematics text book.

First of all, let us look at the important properties of the discriminant.

We know that the roots of the quadratic equation $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{D}}{2a}.$$

Thus the nature of the roots which depends on $D$, can be summarised as follows:

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<th>Discriminant $D$</th>
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In addition, in the quadratic equation $ax^2 + bx + c = 0$, if $a$, $b$ and $c$ are rational numbers, we have:

(a) two roots are rational number if and only if $D$ is a perfect square.

(b) two roots are irrational if and only if $D$ is not a perfect square and $D > 0$.

We can also establish two important properties of a quadratic equation as follows:

$$f(x) = ax^2 + bx + c = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}.$$

Since $\left(x + \frac{b}{2a}\right)^2$ is always non-negative for all real $x$, we have:

$$f(x) \geq c - \frac{b^2}{4a} \quad \text{(i.e. } f \text{ has a minimum value)}$$

$$f(x) \leq c - \frac{b^2}{4a} \quad \text{(i.e. } f \text{ has a maximum value)}$$

Thus $f(x) \geq c - \frac{b^2}{4a}$ for all $x$ if and only if $D < 0$ and $a > 0$. Similarly, $f(x) < c$ for all real $x$ if and only if $a < 0$ and $D < 0$. Hence, two very useful properties of a quadratic function can be summarised as follows:

- $f(x) \geq 0$ for all real $x$ if and only if $a > 0$ and $D \leq 0$
- $f(x) \leq 0$ for all real $x$ if and only if $a < 0$ and $D \leq 0$

We shall explore some applications of these two properties by using the following examples.

### Finding an upper bound

**Example 1:** $A$, $B$ and $C$ are the interior angles of a triangle $ABC$.

Find an upper bound for $\cos\frac{A-B}{2} \cos\frac{A+B}{2} - \cos^2\frac{A+B}{2}$

**Solution:** We have $A + B + C = \pi$.

Let $y = -\cos^2\frac{A+B}{2} + \cos\frac{A-B}{2} \cos\frac{A+B}{2}$.

Re-arranging, $\cos^2\frac{A+B}{2} - \cos\frac{A-B}{2} \cos\frac{A+B}{2} + y = 0$.

Treating this equation as a quadratic equation in $\cos\frac{A+B}{2}$ and since $\cos\frac{A+B}{2}$ is real, we have

$$D = \cos^2\frac{A-B}{2} - 4y \geq 0.$$

Thus $y \leq \frac{1}{4} \cos^2\frac{A-B}{2} \leq \frac{1}{4}$. Hence the upper bound for

$$\cos^2\frac{A-B}{2} \cos\frac{A+B}{2} - \cos^2\frac{A+B}{2}$$

is $\frac{1}{4}$. Note that with this upper bound, we can prove easily that:

$$\sin^2\frac{A}{2} \sin^2\frac{B}{2} \sin^2\frac{C}{2} \leq \frac{1}{8},$$

where $A$, $B$ and $C$ are angles of a triangle.

Let $y = \frac{1}{2} \cos\frac{A-B}{2} - \cos\frac{A+B}{2} \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$.

Then $y = \frac{1}{2} \cos\frac{A-B}{2} - \cos\frac{A+B}{2} \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} \leq \frac{1}{8}$.

### Proving inequalities

**Example 2:** If $x$, $y$ and $z$ are real numbers, prove that $x^2 - xz + z^2 + 3y(x + y - z) \geq 0$.

**Solution:** Let $f(x) = x^2 - xz + z^2 + 3y(x + y - z) = x^2 + x(3y - z) + 3y(y - z) + z^2$.

Treat $f$ as a quadratic function in $x$ and we check its discriminant.

$$D = (3y - z)^2 - 4(3y^2 - 3yz + z^2) = -3y^2 + 6yz - 3z^2 = -3(y - z)^2 \leq 0.$$

As the coefficient of $x^2$ is 1, we can conclude that $f(x) \geq 0$ for all real $x$. Hence, $x^2 - xz + z^2 + 3y(x + y - z) \geq 0$. 

Determining the nature of a triangle

Example 3: A, B and C are the interior angles of a triangle ABC. If \( \cot A + \cot B + \cot C = \sqrt{3} \), determine the nature of this triangle.

Solution: We have

\[
\cot C = \cot((\pi - (A + B))) = \cot(A + B) = \frac{1 - \cot A \cot B}{\cot A + \cot B}
\]

Substitute into the given condition, we have,

\[
\cot A + \cot B + \frac{1 - \cot A \cot B}{\cot A + \cot B} = \sqrt{3}
\]

Let \( a = \cot A, b = \cot B \) and \( c = \cot C \) and then \( a, b \) and \( c \) are real numbers. We have:

\[
a + b + \frac{1 - ab}{a + b} = \sqrt{3},
\]

\[
a^2 + (b - \sqrt{3})a + (b^2 - \sqrt{3}b + 1) = 0.
\]

This is a quadratic equation in \( a \) and \( a \) is a real, its discriminant must be non-negative. Now

\[
D = (b - \sqrt{3})^2 - 4(b^2 - \sqrt{3}b + 1)
\]

\[
= -3b^2 + 2\sqrt{3}b - 1 = -(\sqrt{3}b - 1)^2.
\]

Hence we must have \((\sqrt{3}b - 1)^2 = 0\). Thus \( b = \frac{1}{\sqrt{3}} \). As \((1)\) is symmetric in \( a \) and \( b \), we should have \( a = \frac{1}{\sqrt{3}} \) also. Hence, \( A = B = \frac{\pi}{3} \) and triangle \( ABC \) is equilateral.

Solving Equations

Example 4: (1983 Suzhou Secondary Schools Mathematics Competition)

Find real \( x \) such that \( A = \frac{x^2 - 2x + 4}{x^2 - 3x + 3} \) is an integer.

Solution: \( A = \frac{x^2 - 2x + 4}{x^2 - 3x + 3} = 1 + \frac{x + 1}{x^2 - 3x + 3} \). We need to find real \( x \) such that \( A = \frac{x + 1}{x^2 - 3x + 3} \) is an integer. Cross multiplying, we have

\[ax^2 - (3a + 1)x + 3a - 1 = 0.\]

As \( x \) is real, \( D = (3a + 1)^2 - 4a(3a - 1) \) \( \geq 0 \), so \( 3a^2 - 10a - 1 \leq 0 \). This gives \( \frac{5 - 2\sqrt{7}}{3} \leq a \leq \frac{5 + 2\sqrt{7}}{3} \). Since \( a \) is an integer, \( a \) can take values 0, 1, 2 or 3. Substituting the values of \( a \) back into the above quadratic equation, we can solve for \( x \) which is \(-1, 2 \pm \sqrt{2}, \frac{5}{2} \) and \( \frac{7}{2} \). We can check that these \( x \) values produce an integer \( A \).

Determining the nature of roots

Example 5: Suppose that a quadratic equation \( ax^2 + bx + c = 0 \) has real roots. Show that if \( a, b \) and \( c \) are odd, then the roots are irrational.

Proof: It suffices to prove that \( D \) is not a perfect square. As \( a, b \) and \( c \) are given to be odd, so \( D = b^2 - 4ac \) is an odd number. The square of an odd number is of the form \( 8k + 1 \) as \((2n + 1)^2 = 4n^2 + 4n + 1 = 4n(n + 1) + 1 = 8k + 1 \).

Let \( a = 2m + 1, b = 2n + 1 \) and \( c = 2r + 1 \), so

\[
D = (2n + 1)^2 - 4(2m + 1)(2r + 1)
\]

\[
= 4n(n + 1) + 1 - 4(4mr + 2(m + r) + 1)
\]

\[
= 4n^2 + 4n + 1 - 8mr - 4(m + r) - 3
\]

which is not in the form of \( 8k + 1 \). Hence shown.

Conclusion

This note has shown that the discriminant can be a very useful tool in solving some mathematics problems. We hope that these examples will inspire the students to better understand and apply the discriminant and its properties.

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