contest

1. Prizes in the form of book vouchers will be awarded to the first received best solution(s) submitted by secondary school or junior college students in Singapore for each of these problems.

2. To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

3. Solutions should be typed and sent to : The Editor, Mathematics Medley, c/o Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore117543; and should arrive before 31 January 2000.

4. The Editor's decision will be final and no correspondence will be entertained.

Problems Corner

Problem 1

Let x_1, x_2, x_3, x_4 denote the four roots of the equation

$$x^4 - 18x^3 + kx^2 + 90x - 2000 = 0$$

where k is a constant. If $x_1x_2 = 50$, find the value of k.

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Problem 2

For each positive integer n, let A_n be the (unique) positive integer which satisfies

$$\left(\sqrt{3}+1\right)^{2n} \le A_n < \left(\sqrt{3}+1\right)^{2n} + 1.$$

Prove that A_n is divisible by 2^{n+1}.

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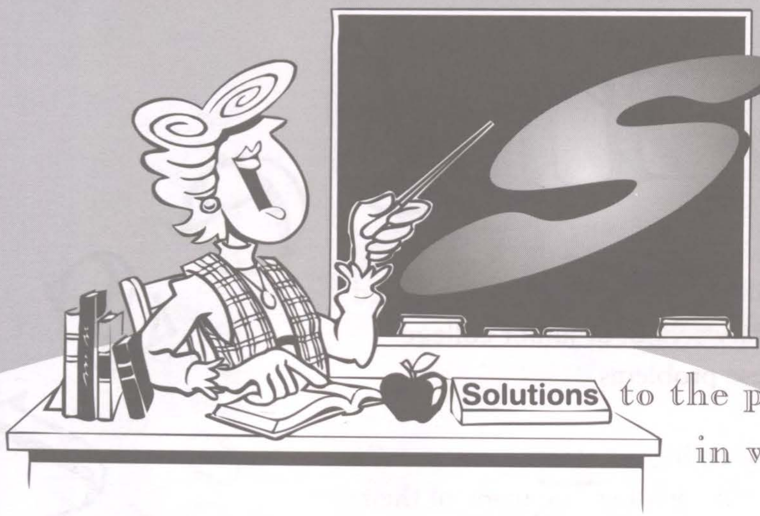

Solutions

Solutions to the problems in volume 26 No. 1 July 1999.

Problem 1

Let M denote the mid-point of the side BC in a triangle ABC. A straight line intersects AB, AM, AC at D, E, F respectively where D lies between A and B and F lies between A and C. Prove that

$$\frac{AM}{AE}=\frac{1}{2}\left(\frac{AC}{AF}+\frac{AB}{AD}\right).$$

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Solution by

Lim Chong Jie
Temasek Junior College
Class 05/98.

Let A be the origin, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{m}$ be the position vectors of the points B, C, D, E, F and M respectively, then $\mathbf{m}=\frac{1}{2}(\mathbf{b}+\mathbf{c})$.

Since D is on AB, let $\mathbf{d}=\lambda\,\mathbf{b}$. Similarly, let $\mathbf{e}=\mu\,(\mathbf{b}+\mathbf{c})$ and $\mathbf{f}=\alpha\,\mathbf{c}$.

Since D, E and F are collinear, $\mathbf{e}=\mathbf{d}+\beta\,(\mathbf{f}-\mathbf{d})=\lambda\,\mathbf{b}+\beta\,(\alpha\,\mathbf{c}-\lambda\,\mathbf{b})=\lambda\,(1-\beta)\,\mathbf{b}+\beta\alpha\,\mathbf{c}$.

Compare the two equations for $\mathbf{e}$, we obtain

$$\mu=\frac{\alpha\lambda}{\alpha+\lambda} \quad\text{and}\quad \beta=\frac{\lambda}{\alpha+\lambda}.$$

Hence, $\dfrac{AM}{AE}=\dfrac{1}{2\mu}=\dfrac{\alpha+\lambda}{2\alpha\lambda}=\dfrac{1}{2}\left(\dfrac{1}{\lambda}+\dfrac{1}{\alpha}\right)=\dfrac{1}{2}\left(\dfrac{AB}{AD}+\dfrac{AC}{AF}\right).$

Solved also by Lu Shang Yi, Raffles Junior College, Class 2SO1C; Tan Eng Chwee, Anderson Secondary School, Class 5/1; Sun Zhao, Nanyang Girls' High School, Class 3/2; and Li Guang, Nanyang Girls' High School, Class 3/10. One incorrect solution was received.

editor's note

The prize was shared equally between Lim Chong Jie and Lu Shang Yi.

Problem 2

Take any 1999 real numbers $x_1, x_2, \ldots, x_{1999}$ such that $0 \le x_n \le 1$ for all $n = 1, \ldots, 1999$. Prove that

$$\left(\frac{1}{1999}\sum_{n=1}^{1999} x_n^2\right) - \left(\frac{1}{1999}\sum_{n=1}^{1999} x_n\right)^2 \le \frac{999000}{1999^2}$$

and determine when will equality hold.

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Solution by the Editor

We shall prove the required inequality with "=" holds if and only if either 999 of the x_i's equal to zero and the other 1000 equal to 1 or 999 of the x_i's equal to 1 and the other 1000 equal to zero.

Let $f(x_1,\ldots,x_{1999})$ denote the left hand side of the required inequality. Let us first consider f as a function of x_1 with $x_2,\ldots,x_{1999}$ fixed. We have

$$f(x_1,\ldots,x_{1999}) = \frac{1998}{1999^2} x_1^2 - \left(\frac{2}{1999^2}\sum_{i=2}^{1999} x_i\right) x_1 + \left[\frac{1}{1999}\sum_{i=2}^{1999} x_i^2 - \left(\frac{1}{1999}\sum_{i=2}^{1999} x_i\right)^2\right]$$

which is a quadratic function in x_1 with the coefficient of x_1^2 positive. Therefore f as a function of x_1 attains its maximum at either $x_1 = 0$ or $x_1 = 1$ (since $0 \le x_1 \le 1$).

A similar consideration shows that f attains its maximum at either $x_i = 0$ or $x_i = 1$ for all $i = 1,\ldots,1999$. Now we let k denote an integer with $-999 \le k \le 1000$ such that $999 + k$ of the x_i's are zero and the other $1000 - k$ equal to 1. Then we have

$$f = \frac{1}{1999}(1000-k) - \left(\frac{1000-k}{1999}\right)^2 = \frac{999000 - k(k-1)}{1999^2}$$

and hence $f \le \dfrac{999000}{1999^2}$ with "=" if and only if $k = 0$ or $k = 1$.

Two incomplete solutions were received.