Prizes in the form of book vouchers will be awarded to the first received best solution(s) submitted by secondary school or junior college students in Singapore for each of these problems.

To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

Solutions should be typed and sent to: The Editor, Mathematics Medley, c/o Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543; and should arrive before 31 October 2000.

The Editor's decision will be final and no correspondence will be entertained.

Problem 1.
Find all pairs of positive integers \((m, n)\) such that \(m^2 + 2000 = 6^n\).

(Prize One $100 book Voucher)

Problem 2.
ABCD is a quadrilateral inscribed in a circle with centre at \(O\), and \(P\) is the intersection of \(AC\) and \(BD\). Let \(O_1, O_2, O_3\) and \(O_4\) be the circumcentres of the triangles \(PAB, PBC, FCD\) and \(PDA\) respectively. Prove that the lines \(OP, O_1O_2, O_3O_4\) are concurrent.

(Prize One $100 book Voucher)
Problem 1

Let \( x_1, x_2, x_3, x_4 \) denote the four roots of the equation

\[ x^4 - 18x^3 + kx^2 + 90x - 2000 = 0 \]

where \( k \) is a constant. If \( x_1x_2 = 50 \), find the value of \( k \).

The answer is \( k = 87 \).

From the given equation, we obtain the following equations:

\[ x_1 + x_2 + x_3 + x_4 = 18 \quad (1) \]
\[ x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = k \quad (2) \]
\[ x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = -90 \quad (3) \]
\[ x_1x_2x_3x_4 = -2000 \quad (4) \]

Given \( x_1x_2 = 50 \), by (4), \( x_3x_4 = -40 \).

Therefore (3) becomes \( 50(x_3 + x_4) - 40(x_1 + x_2) = -90 \).

From (1) & (3), we obtain \( x_1 + x_2 = 11 \) and \( x_3 + x_4 = 7 \).

From (2), \( x_1x_2 + (x_1 + x_2)(x_3 + x_4) + x_3x_4 = k \).

Substituting the known values, we obtain \( k = 87 \).

Solved also by Tan Tze Jwee Glynn, Anglo-Chinese Junior College, Class 2SA1; S. Thiagarajah: Lim Yin, Victoria Junior College, Class 99S22; Julius Poh Wei Quan, AngloChinese (Independent) School, Class 4.14; and Zhang Nan Ruo, Kranji Secondary School, Class 4D.

Three incorrect solutions were received.

Editor's note:
The prize was shared equally between Lim Chong Jie and Tan Tze Jwee Glynn.
For each positive integer $n$, let $A_n$ be the (unique) positive integer which satisfies

$$(\sqrt{3} + 1)^{2n} \leq A_n < (\sqrt{3} + 1)^{2n+1}. \tag{1}$$

Prove that $A_n$ is divisible by $2^{n+1}$.

### Lemma 1

For each positive integer $n$, $B_n$ is a positive integer.

**Proof.**

$$B_n = (\sqrt{3} + 1)^{2n} + (\sqrt{3} - 1)^{2n}$$

$$= \sum_{k=0}^{2n} \binom{2n}{k} (\sqrt{3})^k + \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (\sqrt{3})^k$$

$$= 2 \sum_{k=0}^{n} \frac{2n}{2k} (\sqrt{3})^k$$

$\therefore B_n$ is a positive integer.

### Lemma 2

For each positive integer $n$, we have

$$A_n = B_n.$$

**Proof.**

$$0 < (\sqrt{3} - 1)^{2n} < 1$$

$$0 < (\sqrt{3} - 1)^{2n} < 1$$

$$(\sqrt{3} + 1)^{2n} < B_n < (\sqrt{3} + 1)^{2n+1}$$

$$-1 < B_n - A_n < 1$$

$\therefore$ $B_n$ is an integer by Lemma 1 and $A_n$ is given to be an integer

$\therefore B_n - A_n$ is also an integer and hence

$$B_n - A_n = 0$$

$\therefore A_n = B_n.$
Lemma 3. For each positive integer \( n \), we have

\[ A_{n+2} = 8A_{n+1} - 4A_n. \]

Proof. By Lemma 2, we have

\[
A_n = B_n = (\sqrt{3} + 1)^n + (\sqrt{3} - 1)^n = (4 + 2\sqrt{3})^n + (4 - 2\sqrt{3})^n.
\]

Let \( p = 4 + 2\sqrt{3}, \ q = 4 - 2\sqrt{3}. \)

\[ p + q = 8 \quad \text{and} \quad pq = 4. \]

\[ p \text{ and } q \text{ are the roots of the quadratic equation } x^2 - 8x + 4 = 0. \]

\[ p^2 = 8p - 4 \quad \text{and} \quad q^2 = 8q - 4. \]

\[ p^{n+2} = 8p^{n+1} - 4p^n \quad \text{and} \quad q^{n+2} = 8q^{n+1} - 4q^n. \]

i.e. \( A_{n+2} = 8A_{n+1} - 4A_n. \)

We shall now prove by induction that \( A_n \) is divisible by \( 2^{n+1} \).

We have \( A_1 = 8 \) and \( A_2 = 56 \) and they are divisible by \( 2^{1+1} = 4 \) and \( 2^{2+1} = 8 \) respectively. Now assume that for each \( k \leq n \), we have \( 2^{k+1} \) divides \( A_k \).

\[ A_n = 2^{n-1}x \quad \text{and} \quad A_{n-1} = 2^n y \quad \text{for some positive integers } x \text{ and } y. \]

\[ \text{By Lemma 3, we have} \]

\[ A_{n+1} = 8A_n - 4A_{n-1} = 8(2^{n-1}x) - 4(2^n y) = 2^n(4x - y). \]

\[ A_{n+1} \text{ is divisible by } 2^{n+2}. \]

\[ \text{By induction, } A_n \text{ is divisible by } 2^{n+1}. \]

One incorrect solution was received.