Letter to the Editor

The Editors
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Dear Editors,

The solution on pp. 25–26 of Math. Medley vol. 27 no. 1 (Aug. 2000) to Problem 2 in the previous issue of MM is quite interesting. One may use a simpler argument, as follows.

$$\begin{aligned}(\sqrt{3}+1)^{2n}+(\sqrt{3}-1)^{2n} &= (4+2\sqrt{3})^n+(4-2\sqrt{3})^n\\ &= 2^n[(2+\sqrt{3})^n+(2-\sqrt{3})^n]\\ &= 2^n\Big[\sum_{k=0}^{n}\binom{n}{k}2^{n-k}(\sqrt{3})^k+\sum_{k=0}^{n}\binom{n}{k}2^{n-k}(-1)^k(\sqrt{3})^k\Big]\\ &= 2^{n+1}\sum_{h=0}^{m}\binom{n}{2h}2^{n-2h}3^h,\end{aligned}$$

which is a positive integer, B_n, divisible by 2^{n+1}. Here m is the non-negative integer for which $n=2m$ in the case that n is even and $n=2m+1$ in the case that n is odd.

Now $(\sqrt{3}-1)$ is a positive number less than 1 and thus so is $(\sqrt{3}-1)^{2n}$. Therefore the integer B_n just obtained must be the same as the unique positive integer A_n satisfying

$$(\sqrt{3}+1)^{2n} \le A_n < (\sqrt{3}+1)^{2n}+1,$$

as wanted.

In fact when n is odd, A_n is divisible by 2^{n+2}, as can be seen from the proof above.

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