Intergration of x^n for all values of n

by

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People in the field of calculus should be familiar with the result

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C,$$

for all values of n except $n = -1$. Generally it is readily accepted. With a little advance into the subject, they soon see that $\int \frac{1}{x} dx = \ln x + C$. That completes for the integration of x^n. I myself had felt that the exceptional case for $n = -1$ is really very special, different and outstanding. Why is a $\ln x + C$ among a host of $\frac{x^{n+1}}{n+1} + C$, it is very alien! I suppose this feeling is natural and common. So goes my exploration.

Examining the graphs of $y = \frac{x^{n+1}}{n+1}$ for different values of n, for instance, $n = 0, 1, 2, -0.5, -0.9, -0.99$, clearly shows that the graph of $y = \ln x$ is very much apart from the graphs of $y = \frac{x^{n+1}}{n+1}$ for all $n \neq -1$. The graph of $y = \frac{x^{n+1}}{n+1}$ is not approaching to the graph of $y = \ln x$ when n tends to -1 as we might wish that it would. Let us watch the similarity of the graphs of $y = \frac{x^{0.01}}{0.01}$ and $y = \ln x$. See Figure 1 and 2.

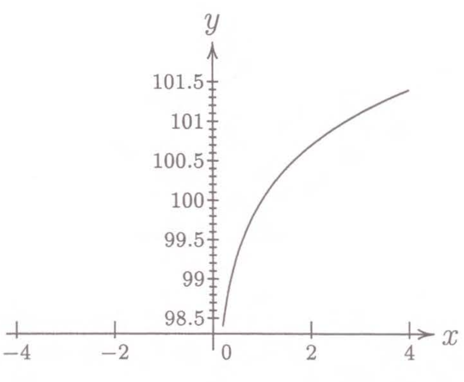

Figure 1 Graph of $y = \frac{x^{0.01}}{0.01}$

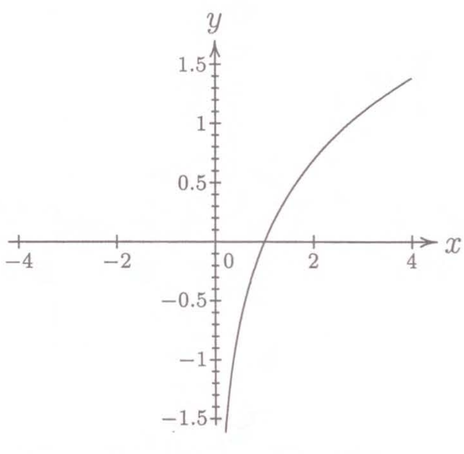

Figure 2 Graph of $y = \ln x$

Their shapes are close! In fact, $\dfrac{d}{dx}(\dfrac{x^{0.01}}{0.01}) = x^{-0.99}$ whereas $\dfrac{d}{dx}\ln x = \frac{1}{x} = x^{-1}$. Indeed their gradients are close! The two curves are different mainly by being far displaced from each other. Consider now the graphs of $y = \dfrac{x^{0.01} - 1}{0.01}$ and $y = \ln x$.

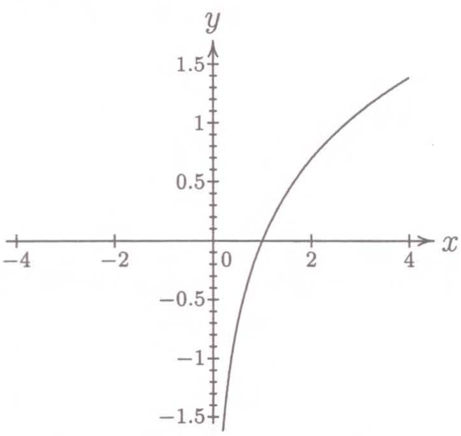

Figure 3 Graph of $y = \frac{x^{0.01}-1}{0.01}$

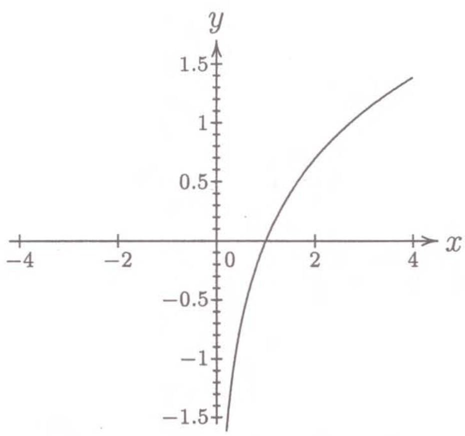

Figure 4 Graph of $y = \ln x$

They are close. Let us adopt a special particular integral:

$$\int x^n dx = \frac{x^{n+1} - 1}{n+1} + C.$$

Examine the graphs of $y = \dfrac{x^{n+1} - 1}{n+1}$ now for different values of n as we do for $y = \dfrac{x^{n+1}}{n+1}$. In particular, consider the graph of $y = \dfrac{x^{-0.99+1} - 1}{-0.99+1}$. See Figure 5.

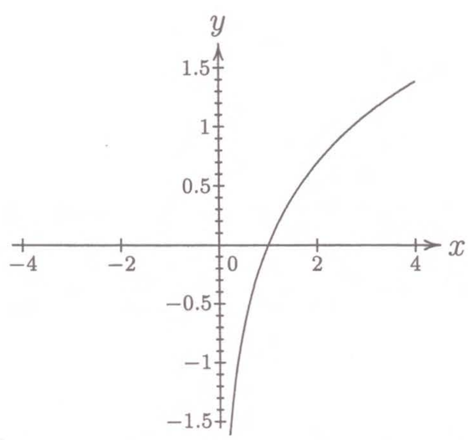

Figure 5 Graph of $y = \frac{x^{-0.99+1}-1}{-0.99+1}$

We see that the graph of $y = \dfrac{x^{n+1} - 1}{n+1}$ approaches that of $y = \ln x$ as n tends to -1. The underlying fact is: $\lim\limits_{n \to -1} \dfrac{x^{n+1} - 1}{n+1} = \ln x$ as the definition of $\ln x$. Alternatively,

$$\begin{aligned}
\lim_{n \to -1} \frac{x^{n+1} - 1}{n+1} &= \lim_{m \to 0} \frac{x^m - 1}{m} \\
&= \lim_{m \to 0} \frac{e^{(\ln x)m} - 1}{m} \\
&= \lim_{m \to 0} \frac{(\ln x)m + \frac{1}{2!}[(\ln x)m]^2 + \frac{1}{3!}[(\ln x)m]^3 + \cdots}{m} \\
&= \ln x.
\end{aligned}$$

Conclusion As an alternate form of result, we might take

$$\int x^n dx = \frac{x^{n+1} - 1}{n+1} + C,$$

so that $\int \frac{1}{x} dx = \ln x + C$ is just a result expected, coming to scene as n approaches -1, but yet, it cannot be obtained by putting $n = -1$ directly.

Classroom Corner