Classroom Corner

A Methodology Suggested for the Determination of the Second Derivative Value at a Stationary Point

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In this article, we assume the existence of the derivative of any order of a function whenever it requires.

In the course of determination of the nature of a stationary point, the option of finding the second derivative value or its sign at the point could be a daunting one sometimes. However, it might be less so when a possibly existing short-cut is taken.

For demonstration, consider $f(x) = x + \sqrt{a^2 - x^2}$ where $a > 0$

We find that $f'(x) = \dfrac{\sqrt{a^2 - x^2} - x}{\sqrt{a^2 - x^2}}$

$f'(x) = 0$ only for $x = \dfrac{a}{\sqrt{2}}$ in this case

On examining

$$f''(x) = \frac{1}{\sqrt{a^2 - x^2}} \frac{d}{dx}(\sqrt{a^2 - x^2}) + (\sqrt{a^2 - x^2} - x)\frac{d}{dx}\frac{1}{\sqrt{a^2 - x^2}}$$

we see that $(\sqrt{a^2 - x^2} - x)\dfrac{d}{dx}\dfrac{1}{\sqrt{a^2 - x^2}} = 0$ for $x = \dfrac{a}{\sqrt{2}}$,

the evaluation of $\dfrac{d}{dx}\dfrac{1}{\sqrt{a^2 - x^2}}$ is quite wasteful of effort if the finding $f''(\dfrac{a}{\sqrt{2}})$ is the sole aim.

We may proceed as follows:

$$f''(x) = \frac{1}{\sqrt{a^2 - x^2}} \frac{d}{dx}(\sqrt{a^2 - x^2}) + (\sqrt{a^2 - x^2} - x)(......)$$

$$= \frac{1}{\sqrt{a^2 - x^2}}\left(\frac{-x}{\sqrt{a^2 - x^2}} - 1\right) + (\sqrt{a^2 - x^2} - x)(......), \text{ so}$$

$$f''(\frac{a}{\sqrt{2}}) = \phi(\frac{a}{\sqrt{2}}) \text{ where } \phi(x) = \frac{1}{\sqrt{a^2 - x^2}}\left(\frac{-x}{\sqrt{a^2 - x^2}} - 1\right)$$

which is seen to be a negative value

So $x = \dfrac{a}{\sqrt{2}}$ gives the maximum value of $f(x) = x + \sqrt{a^2 - x^2}$

A methodology suggested is thus as follows:

For a function $f(x)$, corresponding to a stationary point $x = x_0$, we can put, in factored form,

$f'(x) = u(x)v(x)$ such that the stationary point $x = x_0$ is given by $u(x) = 0$, that is $u(x_0) = 0$. A simple proof then shows that

$$f''(x_0) = u'(x_0)v(x_0)$$

We need not do a full evaluation of $f''(x)$ to find $f''(x_0)$. The work saved is the evaluation of $\frac{d}{dx}v(x)$

For further demonstration, take $S = \frac{1}{8}a^2 \sec\theta \operatorname{cosec}^3\theta$

We find that $\frac{dS}{d\theta} = \frac{a^2}{8}(-3\operatorname{cosec}^4\theta + \sec^2\theta \operatorname{cosec}^2\theta)$

and that $\frac{dS}{d\theta} = 0$ for $\tan\theta = \sqrt{3}$ or $\tan\theta = -\sqrt{3}$

The course of solving of or the set of solutions for $\frac{dS}{d\theta} = 0$ leads to

$$\frac{dS}{d\theta} = \frac{a^2}{8}(\operatorname{cosec}^4\theta)(-3 + \tan^2\theta)$$

For value of $\frac{d^2S}{d\theta^2}$ at stationary points, that is where $-3 + \tan^2\theta = 0$, we write

$$\frac{d^2S}{d\theta^2} = \frac{a^2}{8}(\operatorname{cosec}^4\theta)\frac{d}{d\theta}(-3 + \tan^2\theta) + (-3 + \tan^2\theta)\ (.........)$$

$$= \frac{a^2}{8}(\operatorname{cosec}^4\theta)(2\tan\theta \sec^2\theta) + (-3 + \tan^2\theta)\ (.........)$$. So

$S''(\theta_0) = \phi(\theta_0)$ where $S''(\theta) = \frac{d^2S}{d\theta^2}$, $\phi(\theta) = \frac{a^2}{8}(\operatorname{cosec}^4\theta)(2\tan\theta \sec^2\theta)$ and

$$-3 + \tan^2\theta_0 = 0$$

Furthermore, if only the sign of $S''(\theta_0)$ is required, it can be read by inspection, which is seen to be positive for $\tan\theta_0 = \sqrt{3}$ and negative for $\tan\theta_0 = -\sqrt{3}$

Work in the differentiation of $\frac{dS}{d\theta}$ had only been mainly on $(-3 + \tan^2\theta)$ for the finding of $S''(\theta_0)$

For a wider application of the methodology, we have the following results and guidlines:

(I) For a function $f(x)$ where $f'(x_0) = f''(x_0) = ... = f^{(r)}(x_0) = 0$, if we can put $f'(x) = u(x)v(x)$ such that $x = x_0$ is a solution to $u(x) = 0$ but not $v(x) = 0$, that is $u(x_0) = 0$ but $v(x_0) \neq 0$, then it can be shown that

$f''(x_0) = u'(x_0)v(x_0)$, $\quad f''(x_0) = 0 \Rightarrow u'(x_0) = 0$,

Taking $f'(x)$ as an initial function if $f''(x_0) = 0$, we have

$f'''(x_0) = u''(x_0)v(x_0)$

Inductively, $f'(x_0) = f''(x_0) = ... = f^{(r)}(x_0) = 0 \Rightarrow f^{(r+1)}(x_0) = u^{(r)}(x_0)v(x_0)$.

(II) For the case of a parametric function given by $y = f(t)$ and $x = g(t)$ such that $y = f(g^{-1}(x))$ is well defined, we have

$$\frac{dy}{dx} = \frac{f'(t)}{g'(t)}$$

If we find t_0 such that $f'(t_0) = 0$ but $g'(t_0) \neq 0$, then, for finding $\frac{d^2y}{dx^2}$ at $t = t_0$, we write

$$\frac{d^2y}{dx^2} = \frac{f''(t)}{g'(t)}\frac{dt}{dx} + f'(t)(.......) \text{ and so}$$

value of $\frac{d^2y}{dx^2}$ at $t = t_0$ is value of $\frac{f''(t)}{g'(t)}\frac{dt}{dx} = \frac{f''(t)}{(g'(t))^2}$ at $t = t_0$

If, in addition, we have $f'(t) = u(t)v(t)$ with $u(t_0) = 0$, then $f''(t_0) = u'(t_0)v(t_0)$, more reduction of work is acquired.

(III) For the case of implicit function, if after we obtain a result such as

$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ and find that $f(x_0, y_0) = 0$ but $g(x_0, y_0) \neq 0$, then it can be shown that

$$\phi''(x_0, y_0) = \frac{f'(x_0, y_0)}{g(x_0, y_0)}, \text{ where } \phi''(x,y) = \frac{d^2y}{dx^2} \text{ and } f'(x,y) = \frac{d}{dx}f(x,y),$$

Noting that if $f(x,y) = u(x,y)\, v(x,y)$ such that $u(x_0, y_0) = 0$, more reduction of work is obtainable

Finally, we note that the easiness which the methodology provides lies on the evasion of full evaluation of the second derivative, for which action it may incur some weakness of the methodology.

To optimise benefits in the application, we need try to put $f'(x) = u(x)v(x)$ in such a way that $u(x) = 0$ gives as many stationary points as possible as required on the particular circumstance, to the extent of not loading on the work of differentiation; on the other hand, any factor of $f'(x)$ not for $f'(x) = 0$ or only for stationary points not in concern will be better contained in $v(x)$