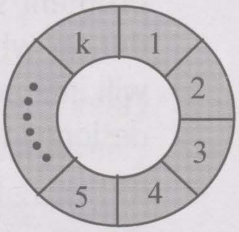
Mathematical Medley
Problems Corner

A Prized Problems

Problem 1

(Book voucher up to $150)

A ring is divided into k sectors (as shown in the diagram). A marble is placed in each sector. In any move, two marbles are shifted, one clockwise and the other anti-clockwise, into the adjacent sectors. (The two marbles being shifted need not come from the same sector.) Is it possible that, after a sequence of moves, all the k marbles end up in the same sector?



Problem 2

(Book voucher up to $150)

Find the smallest integer n such that

$$1=\frac{1}{s_1}+\frac{1}{s_2}+\cdots+\frac{1}{s_n}$$

where s_i are distinct numbers from the arithmetic progression $\{2,5,8,11,14,\ldots\}$.

B Instruction

1. Prizes in the form of book vouchers will be awarded to one or more received best solutions submitted by secondary school or junior college students in Singapore for each of these problems.

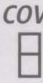
2. To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

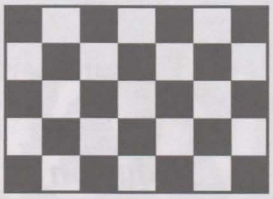
3. Solutions should be sent to :
The Editor, Mathematics Medley,
c/o Department of Mathematics,
National University of Singapore,
2 Science Drive 2, Singapore117543 ;
and should arrive before 1 November 2003.
Alternatively, softcopies of the solutions can also be sent to the email address:
mattanv@nus.edu.sg.

4. The Editor's decision will be final and no correspondence will be entertained.

Solutions to the Problems of Volume 29, No 2, 2002

Problem 1

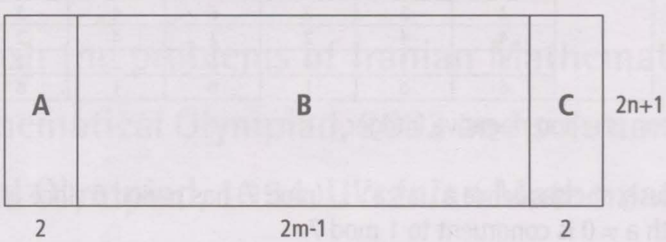
Given a (2m+1) x (2n+1) checkerboard with black squares at the four corners (for example, a 5 x 7 checkerboard is shown in the diagram below), show that if any one white squares and two black squares are removed, the remaining board can be covered with 1 x 2 and 2 x 1 rectangles (i.e.  *and*  *).*



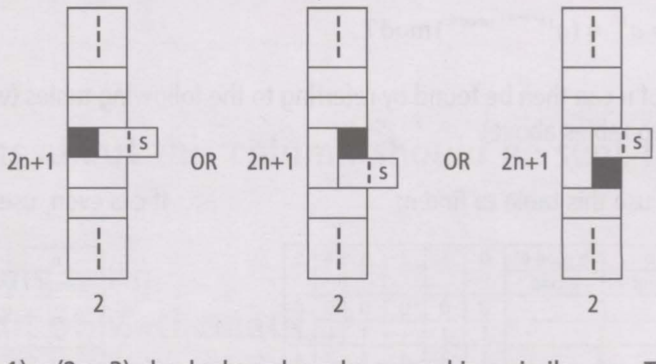
Solution

By Ernest Chong Kai Fong - Raffles Junior College

Note that the case of a 3 x 3 checkerboard can be easily checked to satisfy the conditions by exhaustion. We shall apply induction on this problem. Suppose all checkerboards not larger than (2m+1) x (2n+1) satisfy the conditions. Now consider a (2m+3) x (2n+1) checkerboard. Divide the checkerboard into 3 regions A, B & C as shown in the diagram below:



If region A or C contain none of the removed squares, then induction is immediate, since the removed squares would lie in a (2m+1) x (2n+1) region. Otherwise, we may assume without loss of generality that region A contains 1 of the removed squares while region D contains the other 2 removed squares, where region D is the combinations of regions B & C. Now, by our induction hypothesis, in addition to the 2 removed squares in region D, if we remove any square with the same colour as the removed square in region A, the remaining squares in region D can be covered with the 1 x 2 and 2 x 1 rectangles. We can always choose an appropriate square s in region D that is adjacent to region A such that region A and the square s can be covered by the 1 x 2 and 2 x 1 rectangles as shown below: (Note that the black square represents the removed square in region A)



The case of a (2m+1) x (2n+3) checkerboard can be argued in a similar way. Therefore by induction, a (2m+1) x (2n+1) checkerboard can be covered by the given rectangles for all positive integers m, n.

Editor's note: Solved also by Joel Tay Wei En (Raffles Junior College) and Andre Kueh Ju Lui (Chinese High School). The prize is shared by Ernest, Joel and Andre.

Solutions to the Problems of Volume 29, No 2, 2002

Problem 2

Let a, b and c be positive integers. Suppose today is a Sunday.

What day is it after a^{b^c} days?

Solution

By Sean Lip Zhao Wen - Hwa Chong Junior College

The problem is equivalent to finding the value of $n \equiv a^b \bmod 7$, where a, b, c are positive integers. Then, for n = 0, 1, 2, ..., 6, the corresponding days of the week are Sunday, Monday, Tuesday, ..., Saturday.

Let $k = b^c$. Then, for each a mod 7, $a^k \bmod 7$ has a value according to the following table:

		k					
	a^k mod 7	6m + 1	6m + 2	6m + 3	6m + 4	6m + 5	6m + 6
a mod 7	0	0	0	0	0	0	0
	1	1	1	1	1	1	1
	2	2	4	1	2	4	1
	3	3	2	6	4	5	1
	4	4	2	1	4	2	1
	5	5	4	6	2	3	1
	6	6	1	6	1	6	1

where m is a non-negative integer.

Obviously, the sequence $a^1, a^2, a^3, \ldots$ (mod 7) has period 6 since any number of the form a^6 with $a \neq 0$ is congruent to 1 mod 7.

Similarly, we construct a table for b^c (mod 6).

		c			
	b^c mod 6	1	2	3	...
b mod 6	0	0	0	0	
	1	1	1	1	
	2	2	4	2	
	3	3	3	3	
	4	4	4	4	
	5	5	1	5	

Notice that the sequence $b^1, b^2, b^3, \ldots$, has period 2.

Hence, $n = a^{b^c} \equiv (a^{(b^{c \bmod 2}) \bmod 6}) \bmod 7$.

The value of n can then be found by referring to the following tables (which are produced by combining the first two tables above)

If c is odd, use this table to find n:

n / a mod 7	b mod 6 / c odd	0	1	2	3	4	5
0		0	0	0	0	0	0
1		1	1	1	1	1	1
2		1	2	4	1	2	4
3		1	3	2	6	4	5
4		1	4	2	1	4	2
5		1	5	4	6	2	3
6		1	6	1	6	1	6

If c is even, use this table to find n:

n / a mod 7	b mod 6 / c even	0	1	2	3	4	5
0		0	0	0	0	0	0
1		1	1	1	1	1	1
2		2	2	2	1	2	2
3		3	3	4	6	4	3
4		4	4	4	1	4	4
5		5	5	2	6	2	5
6		6	6	1	6	1	6

Editor's note: Solved also by Ernest Chong Kai Fong. The prize is shared by Sean and Ernest.