Classroom Corner

About Removal of Modulus Sign From |f(x)|>g(x)

Chan Wei Min

It is a well-known fact that

$$|f(x)| > a \tag{i}$$

is equivalent to

$$f(x) > a \text{ or } f(x) < -a. \tag{ii}$$

Similarly,

$$|f(x)| < a \tag{iii}$$

is identical to

$$-a < f(x) < a \tag{iv}$$

Though, in the above, it might be thought that it is meaningful only if $a > 0$, yet it is not so.

In fact, if $a < 0$,

(i) qualifies all x for which $f(x)$ is defined, so does (ii). On the contrary,

(ii) disqualifies all x for which $f(x)$ is defined, so does (iv).

If $a = 0$,

(i) qualifies all x for which $f(x)$ is defined, where $f(x) \neq 0$, so does (ii).

(ii) disqualifies all x for which $f(x)$ is defined, so does (iv).

We thus have "The Statement":

(i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) for all a.

At this point, one would naturally like to see whether there are alike results for $|f(x)| > g(x)$, as well as for $|f(x)| < g(x)$. Indeed, The Statement above should have justified that

$$|f(x)| > g(x) \tag{v}$$

is the same as

$$f(x) > g(x) \text{ or } f(x) < -g(x) \tag{vi}$$

and

$$|f(x)| < g(x) \tag{vii}$$

is the same as

$$-g(x) < f(x) < g(x). \qquad \text{(viii)}$$

Just to elaborate a little more, perhaps for a better insight, we shall discuss about
(v) $\Leftrightarrow$ (vi).

(I) With due consideration given to the sign of $g(x)$, let $A = \{x : g(x) \geq 0\}$. For $x \in A$, $|f(x)| > g(x)$ is the same as

$$\text{“}f(x) > g(x) \text{ or } f(x) < -g(x)\text{”}.$$

For $x \notin A$, $|f(x)| > g(x)$ is true for all x, so is

$$\text{“}f(x) > g(x) \text{ or } f(x) < -g(x)\text{”}.$$

We conclude that (v) $\Leftrightarrow$ (vi) for whatever sign of $g(x)$ inclusive of zero.

For an insight into how sets of values in the course of solving for (v) converge to that for (vi), let

$$P = \{x : f(x) > g(x)\}, \qquad Q = \{x : -f(x) > g(x)\}.$$

The solution set for (v) is $(A \cap P) \cup (A \cap Q) \cup A'$ and the solution set of (vi) is $P \cup Q$. We have

$$\begin{aligned}
&((A \cap P) \cup (A \cap Q)) \cup A' \\
&= ((A \cap P) \cup (A \cap Q)) \cup (A' \cap (P \cup Q)) \quad \text{as } A' \subset P \cup Q \\
&= (A \cap (P \cup Q)) \cup (A' \cap (P \cup Q)) \\
&= P \cup Q.
\end{aligned}$$

(II) With due consideration given to the sign of $f(x)$. Let

$$B = \{x : f(x) \geq 0\}.$$

For $x \in B$,

$$|f(x)| > g(x) \Leftrightarrow f(x) > g(x) \Leftrightarrow f(x) > g(x)$$

or $$f(x) < -g(x)$$

as $$f(x) < -g(x) \Rightarrow g(x) \text{ is negative } \Rightarrow f(x) > g(x).$$

For $x \notin B$,

$$|f(x)| > g(x) \Leftrightarrow -f(x) > g(x) \Leftrightarrow -f(x) > g(x)$$

or $$-f(x) < -g(x),$$

as $$-f(x) < -g(x) \Rightarrow g(x) \text{ is negative } \Rightarrow -f(x) > g(x).$$

We conclude that (v) $\Leftrightarrow$ (vi) for whatever sign of $f(x)$ inclusive of zero. For an insight into how sets of values from (v) converge to that from (vi), the solution set for (v) is $(B \cap P) \cup (B' \cap Q)$ and the solution of (vi) is $P \cup Q$. We have

$$\begin{aligned}(B \cap P) \cup (B' \cap Q) &= ((B \cap P) \cup (B \cap Q)) \cup ((B' \cap P) \cup (B' \cap Q)) \\ &\qquad \text{as } B \cap Q \subset B \cap P \text{ and } B' \cap P \subset B' \cap Q \\ &= (B \cap (P \cup Q)) \cup (B' \cap (P \cup Q)) \\ &= P \cup Q.\end{aligned}$$

A declaimer.

In solving such inequality as $|f(x)| > g(x)$, an immediate conversion to "$f(x) > g(x)$ or $f(x) < -g(x)$" should prove to be effective. Yet, this could be seen as an easy deduction when the sign of $g(x)$ is overlooked. We suggested that it be backed by the basic truth that

$$|b| > a \Leftrightarrow \text{"}b > a \text{ or } b < -a\text{" for all } a.$$

404 Pandan Gardens #12-24,
Singapore 600404
chanwmts@hotmail.com