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**PROBLEMS**

1. (Bulgaria 1994) A triangle $ABC$ is given with $p = \frac{1}{2}(AB + BC + CA)$. A circle $k_1$ touches the side $BC$ and the sides $AB$, $AC$ extended. A circle $k$ touches $k_1$ and the incircle of $\triangle ABC$ at points $Q$ and $P$. Let $R$ be the point of intersection of the line $PQ$ and the bisector of $\angle BAC$ and $RT$ be a tangent to $k$. Prove that $RT = \sqrt{p(p-a)}$.

2. (Russia 2000) A positive integer $n$ is called perfect if the sum of all its positive divisors excluding $n$ itself, equals $n$. For example 6 is perfect because $6 = 1 + 2 + 3$. Prove that
   (a) if a perfect integer larger than 6 is divisible by 3, then it is also divisible by 9.
   (b) if a perfect integer larger than 28 is divisible by 7, then it is also divisible by 49.

3. (Russia 2000) Circles $\omega_1$ and $\omega_2$ are internally tangent at $N$, with $\omega_1$ larger than $\omega_2$. The chords $BA$ and $BC$ of $\omega_1$ are tangent to $\omega_2$ at $K$ and $M$, respectively. Let $Q$ and $P$ be the midpoints of the arcs $AB$ and $BC$ not containing the point $N$. Let the circumcircles of triangles $BQK$ and $BPM$ intersect at $B$ and $B_1$. Prove that $BPB_1Q$ is a parallelogram.

4. (Taiwan 2000) Let $A = \{1, 2, \ldots, n\}$, where $n$ is a positive integer. A subset of $A$ is connected if it is a nonempty set which consists of one element or of consecutive integers. Determine the greatest integer $k$ for which $A$ contains $k$ distinct subsets $A_1$, $A_2$, $\ldots$, $A_k$, such that the intersection of any two distinct sets $A_i$ and $A_j$ is connected.

5. (Turkey 2000)
   (a) Prove that for each positive integer $n$, the number of ordered pairs $(x, y)$ of integers satisfying
   
   $$x^2 - xy + y^2 = n$$
   
   is finite and divisible by 6.
   (b) Find all ordered pairs $(x, y)$ of integers satisfying
   
   $$x^2 - xy + y^2 = 727.$$  

6. (Vietnam 2000) Two circles $C_1$ and $C_2$ intersect at two points $P$ and $Q$. The common tangent of $C_1$ and $C_2$ closer to $P$ touches $C_1$ at $A$ and $C_2$ at $B$. The tangent to $C_1$ at $P$ intersects $C_2$ at $E$ (distinct from $P$) and the tangent to $C_2$ at $P$ intersects $C_1$ at $F$ (distinct from $P$). Let $H$ and $K$ be two points on the rays $AF$ and $BE$, respectively, such that $AH = AP$, $BK = BP$. Prove that the five points $A, H, Q, K, B$ lie on the same circle.

7. (St. Petersburg 2000) One hundred points are chosen in the coordinate plane. Show that at most $2025 = 45^2$ rectangles with vertices among these points have sides parallel to the axes.
8. (St. Petersburg 2000) Let $AA_1$, $BB_1$, $CC_1$ be the altitudes of an acute triangle $ABC$. The points $A_2$ and $C_2$ on line $A_1C_1$ are such that line $CC_1$ bisects the segment $A_2B_1$ and line $AA_1$ bisects the segment $C_2B_1$. Lines $A_2B_1$ and $AA_1$ meet at $K$, and lines $C_2B_1$ and $CC_1$ meet at $L$. Prove that lines $KL$ and $AC$ are parallel.

9. (Korea 2000) The real numbers $a, b, c, x, y, z$ satisfy $a^2, b^2, c > 0$ and $x^2, y^2, z^2 > 0$. Prove that

$$\frac{a^2x^2}{(by+cz)(bz+cy)} + \frac{b^2y^2}{(cz+ax)(cx+az)} + \frac{c^2z^2}{(ax+by)(ay+bx)} \geq \frac{3}{4}$$

10. (Mongolia 2000) The bisectors of angles $A, B, C$ of $\triangle ABC$ intersect its sides at points $A_1, B_1, C_1$. Prove that if the quadrilateral $BA_1B_1C_1$ is cyclic, then

$$\frac{BC}{AC+AB} = \frac{AC}{AB+BC} - \frac{AB}{BC+AC}.$$
2. Triangle $ABC$ is acute-angled. Triangles $A'BC$, $B'AC$, $C'AB$ are constructed externally on its sides such that

$$\angle A'BC = \angle B'AC = \angle C'BA = 30^\circ; \angle A'CB = \angle B'CA = \angle C'AB = 60^\circ.$$ 

Show that if $N$ is the midpoint of $BC$, then $B'N$ is perpendicular to $A'C'$.

Correct solutions were received from Joel Tay, Ernest Chong, Charmaine Sia (Raffles Junior College), Andre Kueh (Hwa Chong Junior College), Teo Weihao (National Junior College), Kenneth Tay (Anglo-Chinese Junior College) and Robert Pargeter (UK). We resent the solutions of Weihao and Kenneth.

Let $X, Y, Z$ be the reflections of $C, C, A$ about the lines $A'B, AB', BC'$, respectively. Consider the circumcircles of $\triangle AZB$ and $\triangle CXB$ with $P, Q$ as their respective centres. Let $D$ be another of the intersection points. Then we have

$$\angle ADB = \angle CDB = 120^\circ \Rightarrow \angle ADC = 120^\circ.$$ 

Therefore $ADCY$ is cyclic since $\angle ADC + \angle AYC = 180^\circ$. Thus

$$\angle DAC + \angle CAY + \angle DYA = \angle DYC + \angle CAY + \angle DYA = 120^\circ = \angle ADB.$$ 

Hence $B, D, Y$ are collinear. Thus $BY \perp PQ$. But $PQ \parallel A'C'$ (since $BP/BC' = BQ/BA'$) and $BY \parallel B'N$ (since $N, B'$ are the midpoints of $BC, YC$, respectively). So $A'C' \perp B'N$.

3. Find all natural numbers $n$ for which there exist $n$ unit squares in the plane with horizontal and vertical sides such that the obtained figure has at least 3 symmetry axes.
Solved by Joel Tay, Ernest Chong (Raffles Junior College) and Andre Kueh (Hwa Chong Junior College). We present Kueh’s solution.

It is not possible to have 2 parallel axes of symmetry since the number of squares is finite. Since the sides of the squares are either horizontal or vertical, the axes are either horizontal, vertical or make an angle of $45^\circ$ with the horizontal. Thus two of the axes are perpendicular to each other. So $n \equiv 0, 1 \pmod{4}$.

The cells of square chess boards of any dimension are examples of figures that satisfy the requirement for all such $n$.

4. Find all real polynomials $p(x)$ which satisfy

$$p(2p(x)) = 2p(p(x)) + 2(p(x))^2$$

for all $x \in \mathbb{R}$.

Similar solutions by Andre Kueh (Hwa Chong Junior College), Teo Weihao (National Junior College), Joel Tay, Ernest Chong (Raffles Junior College) and Kenneth Tay (Anglo-Chinese Junior College).

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, with $a_n \neq 0$. The terms with the highest degree in $p(2(px))$, $2p(p(x))$, $(p(x))^2$ are, respectively, $2^n a_n^{n+1} x^{n^2}$, $2a_n^{n+1} x^{n^2}$, $2a_n^2 x^{2n}$. So if $n \geq 3$, we must have

$$2^n a_n^{n+1} x^{n^2} > 2a_n^{n+1} x^{n^2}$$

which is impossible. Thus $n \leq 2$. So letting $p(x) = ax^2 + bx + c$, we have

$$2a^2(a-1)x^4 + 4ab(a-1)x^3 + 2(a-1)(2ac + b^2)x^2 + 4bc(a-1)x + c(2ac - 2c - 1) \equiv 0.$$

Thus all the coefficients are 0. From $a^2(a-1) = 0$ we get $a = 0$ or $a = 1$. For $a = 0$, we get $c = -1/2$, $0$ and $b = 0$. For $a = 1$, we get $c = 0$ and no restriction on $b$. Thus

$$p(x) \equiv 0, \quad p(x) \equiv -1/2, \quad p(x) = x^2 + bx$$

are the required polynomials.

5. In $\triangle ABC$ with $AB > AC$, the bisectors of $\angle B$ and $\angle C$ meet the opposite sides respectively at $P$ and $Q$. Let $I$ be the intersection of these bisectors. If $IP = IQ$, determine $\angle A$.

We present the similar solutions of Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Joel Tay, Ernest Chong, Charmaine Sia (Raffles Junior College) and Kenneth Tay (Anglo-Chinese Junior College). Also solved by Robert Pargeter (UK).

Using sine rule on $\triangle AIQ$ and $\triangle API$, we get $\sin \angle AIQ = \sin \angle API$.

If $\angle AIQ = \angle API$, we have $AB = AC$ which is impossible. Thus $\angle AIQ + \angle API = 180^\circ$. This in turn implies that $180^\circ = \angle ABC + \angle QCB + \angle PCB + \angle ACB = 1.5(\angle ABC + \angle ACB)$. So $\angle BAC = 60^\circ$.

6. Find all functions $f : \mathbb{R}\setminus\{0\} \rightarrow \mathbb{R}$ such that

$$xf(x + \frac{1}{y}) + yf(y) + \frac{y}{x} = yf(y + \frac{1}{x}) + xf(x) + \frac{x}{y}$$
for all $x, y \in \mathbb{R}\backslash\{0\}$.

We present the similar solutions of Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Joel Tay, Ernest Chong, Charmaine Sia (Raffles Junior College) and Kenneth Tay (Anglo-Chinese Junior College).

Define $g(x) = f(x) - x$, then after simplification we have

$$xg(x + \frac{1}{y}) + yg(y) = yg(y + \frac{1}{x}) + xg(x)$$  \hspace{1cm} (1)

Set $y = 1$, we get

$$xg(x + 1) + g(1) = g(1 + \frac{1}{x}) + xg(x)$$ \hspace{1cm} (2)

Interchanging $x$ with $1/x$ gives

$$\frac{1}{x}g(1 + \frac{1}{x}) + g(1) = g(x + 1) + \frac{1}{x}g(\frac{1}{x})$$

So

$$g(1 + \frac{1}{x}) = xg(x + 1) + g(\frac{1}{x}) - xg(1)$$ \hspace{1cm} (3)

Equations (2) and (3) give

$$xg(x) + g(\frac{1}{x}) = (x + 1)g(1)$$ \hspace{1cm} (4)

Setting $y = -1$, we get similarly

$$xg(x) - g(\frac{1}{x}) = -g(-1)(x - 1)$$ \hspace{1cm} (5)

Equations (4) and (5) give

$$2xg(x) = Ax + B.$$

So $f(x) = A' + \frac{B'}{x} + x$ where $A', B'$ are constants.

7. Let $AB$ be a diameter of a circle $O$. Suppose that $\ell_a$, $\ell_b$ are tangent lines to $O$ at $A, B$, respectively. $C$ is an arbitrary point on $O$. $BC$ meets $\ell_a$ at $K$. The bisector of $\angle CAK$ meets $CK$ at $H$. $M$ is the midpoint of the arc $CAB$ and $S$ is another intersection point of $HM$ with $O$. $T$ is the intersection of $\ell_b$ and the tangent line to $O$ at $M$. Show that $S, T, K$ are collinear.

No correct solution was received. We present the official solution.
Since $AB$ is a diameter, $\angle ACB = 90^\circ$. Extend $TM$ to meet $\ell_a$ at $D$. Then $\angle TMF = \angle MCB = \angle MBC$. Therefore $TD \parallel BK$. Thus $TDKB$ is a parallelogram. Let $AH$ meet the circle at $E$. Then $\angle EAC = \angle EAK = \angle EBA = \angle ECA$. Therefore $AE = EC$. As $\angle ACH = 90^\circ$, we have $AE = EC = HE$. Since $\angle AEB = 90^\circ$, we have $AB = HB$.

Note that $S, T, K$ are collinear iff

$$\frac{HS}{SM} = \frac{HK}{MT}$$

(1)

Since $AH$ is the bisector of $\angle CAK$, we have

$$\frac{HK}{HC} = \frac{AK}{AC} = \frac{KB}{AB} \Rightarrow HK = \frac{KB \cdot HC}{AB}.$$ 

Considering the power of $H$, we have

$$HS \cdot HM = HC \cdot HB, \Rightarrow HS = \frac{HC \cdot HB}{HM}.$$ 

Thus (1) is equivalent to

$$\frac{HB}{HM \cdot SM} = \frac{KB}{AB \cdot BT}$$

(2)

Since $AB$ is the diameter,

$$SM = AB \sin \angle MBS = AB \sin \angle MHB.$$ 

The latter follows because $\angle MHB = \angle DMH = \angle MBS$. Thus (2) is equivalent to

$$AB \cdot BT = HM \cdot KB \sin MHB \text{ or } [ABT] = [MKB]$$

where $[ \cdot ]$ denote the area. Since $TDKB$ is a parallelogram, we have $[MKB] = [TKB] = [ABT]$ and the proof is complete.

8. Let $k$ be a nonnegative integer and $a_1, a_2, \ldots, a_n$ be integers such that there are at least $2k$ different integers mod $(n + k)$ among them. Prove that there is a subset of $\{a_1, a_2, \ldots, a_n\}$ whose sum of elements is divisible by $n + k$.

Solved by Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Joel Tay, Ernest Chong (Raffles Junior College) and Kenneth Tay (Anglo-Chinese Junior College). We present the solution by Charmaine Sia (Raffles Junior College).

Let $a_1, \ldots, a_{2k}$ be the $2k$ different integers mod $(n + k)$. Consider the sums:

$$\sum_{i=1}^{j} a_i, \quad j = 1, 2, \ldots, n$$

$$a_{2j+1} + \sum_{i=1}^{2j-1} a_i, \quad j = 1, 2, \ldots, k - 1$$

$$a_2$$
There are altogether \( n + k \) sums. If all the sums are distinct mod \( (n + k) \), then one of them is 0 mod \( (n + k) \) and we are done. Now suppose that there are two sums which are congruent mod \( (n + k) \). Note that if two sums have the same number of terms, they cannot be congruent mod \( (n + k) \). The two sums must have different number of terms. In this case, their difference is the sum of a subset and is 0 mod \( (n + k) \).

9. Consider a permutation \( (a_1, a_2, \ldots, a_n) \) of \( \{1, 2, \ldots, n\} \). We call this permutation quadratic if there exists at least one perfect square among the numbers \( a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_n \). Find all natural numbers \( n \) such that every permutation of \( \{1, 2, \ldots, n\} \) is quadratic.

We present solution of Andre Kueh (Hwa Chong Junior College) and Ernest Chong (Raffles Junior College).

If \( n(n + 1)/2 \) is a square, then clearly every permutation of \( \{1, 2, \ldots, n\} \) is quadratic. We shall first characterize such \( n \). Let \( n(n + 1)/2 = m^2 \). So

\[
(2n + 1)^2 - 2(2m)^2 = 1.
\]

Set \( x = 2n + 1, y = 2m \), we get \( x^2 - 2y^2 = 1 \), which is a Pell's equation with primitive roots \( x_0 = 3, y_0 = 2 \). So if \( (x, y) \) is a solution then

\[
x + \sqrt{2}y = (3 + 2\sqrt{2})^m, \quad m = 1, 2, \ldots
\]

Thus

\[
n = \frac{(3 + 2\sqrt{2})^m + (3 - 2\sqrt{2})^m - 2}{4}, \quad m = 1, 2, \ldots
\]

We now show that no other natural numbers have this property. Consider a permutation \( A = (a_1, a_2, \ldots, a_n) \) of \( \{1, 2, \ldots, n\} \) where \( n(n + 1)/2 \) is not a square,. Let \( s_k(A) = \sum_{i=1}^{k} a_i \). Let \( I \) be the identity permutation, i.e., \( a_i = i \) for all \( i \). Then \( s_k(I) = k(k + 1)/2 \). Suppose \( k(k + 1) = m^2 \). Then \( k < 2m \). So

\[
\frac{(k + 1)(k + 2)}{2} = m^2 + k + 1 < m^2 + 2m + 1 = (m + 1)^2.
\]

If \( I' \) is the permutation obtained from \( I \) by interchanging \( k \) and \( k + 1 \), then both \( s_k(I') \) and \( s_{k+1}(I') \) are not perfect squares. Using this procedure, we can construct from \( I \) a permutation which is not quadratic.

10. A cube of size 10, has 1000 blocks each of size 1, 500 of them are white and the others are black. Show that there exists at least 100 unit squares which are the common face of a white block and a black block.

No correct solution was received. We present the official solution.

We call a square a boundary face if it is the common face of a black and a white block. For any two blocks \( A, B \) of different colours, there is a unique path \( P_{AB} \) which begins at \( A \), goes parallel to the \( z, y \) and \( x \) axes in that order and ends at \( B \). Each \( P_{AB} \) passes through at least one boundary face. There are \( 1000 \times 500 = 500000 \) such paths. Thus there are at least 500000 boundary faces, counting repetition.
Consider an arbitrary square \( S \) parallel to the \( xy \) plane. Assume that there are \( k \) blocks below it. For a path \( P_{AB} \) to pass through \( S \), \( A \) must be directly below \( S \) and \( B \) any block above the plane containing \( S \) or \( A \) is directly above \( S \) and \( B \) any block below the plane containing \( S \). In the former, there are \( k \times (100 \times (10 - k)) \) paths. In the latter, there are \((10 - k) \times (100 \times k) \) paths. Thus each boundary face is counted \( 200k(10 - k) \leq 5000 \) times. Thus the number of boundary faces is \( \geq \frac{500000}{5000} = 100 \).

**Slovenia Mathematical Olympiad, 2002**

Selected problems from the final round.

1. Let \( K \) be a circle in the plane, and \( K_1, K_2 \) be two disjoint circles inside \( K \) and touching \( K \) at \( A, B \), respectively. Let \( t \) be the common tangent line of \( K_1 \) and \( K_2 \) at \( C \) and \( D \), respectively such that \( K_1, K_2 \) are on the same side of \( t \) while the centre of \( K \) is on the opposite side. Denote by \( E \) the intersection of lines \( AC \) and \( BD \). Prove that \( E \) lies on \( K \).

Solved by Robert Pargeter (UK), Charmaine Sia, Ong Xing Cong, Ernest Chong, Joel Tay (Raffles Junior College), Andre Kueh (Hwa Chong Junior College), Teo Weihao (National Junior College). We present the similar solutions of Joel, Charmaine, Andre and Weihao.

Let \( h \) be the homothety with centre \( A \) that maps \( K_1 \) to \( K \). Extend \( AC \) to meet \( K \) at \( E' \). Then \( h \) maps \( t \) to \( t' \) which is the tangent of \( K \) at \( E' \). Note that \( t \parallel t' \).

Extend \( BD \) to meet \( K \) at \( E'' \). Similarly, the homothety \( h' \) with centre \( B \) that takes \( K_2 \) to \( K \) also take \( t \) to \( t'' \), the tangent of \( K \) at \( E'' \). Since \( t \parallel t'' \), we have \( t' = t'' \). Thus \( E' = E'' = E \).

(Note: Pargeter pointed that the location of \( O \) with respect to \( t \) is not important. The result remains true if all the three centres are on the same side of \( t \). This is also evident from the solution given.)

2. There are \( n \geq 3 \) sheets, numbered from 1 through \( n \). The sheets are then divided into two piles and the task is to find out if in at least one pile two sheets can be found such that the sum of the corresponding numbers is a perfect square. Prove that
(a) if \( n \geq 15 \), then two such sheets can always be found;

(b) if \( n \leq 14 \), then there is a division in which two such sheets cannot be found.

_Solved by Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Charmaine Sia, Ong Xing Cong, Ernest Chong, Joel Tay (Raffles Junior College), Kenneth Tay (Anglo-Chinese Junior College). We present their similar solutions._

(a) Assume, on the contrary, that the sum of any pair of numbers in any of the piles is not a perfect square. Let the piles be \( A, B \). Without loss of generality, assume that \( 1 \in A \). Then

\[
1 \in A \Rightarrow 3 \in B \Rightarrow 6 \in A, \Rightarrow 10 \in B \Rightarrow 15 \in A.
\]

This leads to a contradiction.

(b) The following is an example.

\[
A = \{1, 2, 4, 6, 9, 11, 13\}, B = \{3, 5, 7, 8, 10, 12, 14\}.
\]

3. Does there exist a function \( f : \mathbb{N} \to \mathbb{N} \) such that

\[
f(f(2002)) = 17, \quad f(mn) = f(m)f(n), \quad \text{and} \quad f(n) \neq n
\]

for every \( m, n \in \mathbb{N} \)?

_Solved by Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Charmaine Sia, Ong Xing Cong, Ernest Chong, Joel Tay (Raffles Junior College), Kenneth Tay (Anglo-Chinese Junior College). We present their similar solutions._

Since \( 2002 = 2 \times 7 \times 11 \times 13 \), we have

\[
f(f(2002)) = f(f(2)f(7)f(11)f(13)) = f(f(2))f(f(7))f(f(11))f(f(13)) = 17.
\]

Thus \( f(f(2)), f(f(7)), f(f(11)), f(f(13)) \) is a permutation of 1,1,1,17. But \( f(f(x)) \leq f(x) \leq x \). So the above cannot happen. Thus the function cannot exists.

4. Let \( S = \{a_1, \ldots, a_n\} \) where \( a_i \) are different positive integers. The sum of all numbers of any proper subset of the set \( S \) is not divisible by \( n \). Prove that the sum of all numbers of the set \( S \) is divisible by \( n \).

_Solved by Andre Kueh (Hwa Chong Junior College), Charmaine Sia, Ong Xing Cong, Ernest Chong, Joel Tay (Raffles Junior College), Kenneth Tay (Anglo-Chinese Junior College). We present their similar solutions._

Let \( S_i = a_1 + \cdots + a_i, i = 1, \ldots, n \). For \( 1 \leq i < j \leq n \), \( S_j - S_i \) is the sum of a proper subset of \( S \). Thus \( S_j - S_i \neq 0 \pmod{n} \). Hence \( S_i, \ldots, S_n \) are distinct \( \pmod{n} \) and so one of them is divisible by \( n \). Since \( S_i \neq 0 \pmod{n} \) for \( 1 \leq i \leq n-1 \), we have \( n \mid S_n \).

5. Let \( A' \) be the foot of the altitude on the side \( BC \) of \( \triangle ABC \). The circle with diameter \( AA' \) intersects the side \( AB \) at the points \( A \) and \( D \), and the side \( AC \) at the
points $A$ and $E$. Prove that the circumcentre of $\triangle ABC$ lies on the line determined by the altitude on the side $DE$ of $\triangle ADE$.

Solved by Charmaine Sia (Raffles Junior College), Teo Weihao (National Junior College) and Robert Pargeter (UK). We present the similar solutions of Weihao and Charmaine.

Let $F$ be the foot of the altitude from $A$ onto $DE$, $O$ be the midpoint of $AA'$ and $O'$ be the circumcentre of $\triangle ABC$. To prove that $O'$ lies on $AF$, we merely need to show that $\angle BAF = 90^\circ - \angle ACB$. This follows readily from $\angle ADE = \angle AA'E = \angle ACB$.

6. In the cellar of a castle 7 dwarfs protect their treasure. The treasure is behind 10 doors and every door has 3 locks. All locks are different from each other. Every dwarf has the keys for some locks. Any four dwarfs together have keys for all the locks. Prove that there exist three dwarfs who together have the keys for all the locks.

Solved by Teo Weihao (National Junior College), Andre Kueh (Hwa Chong Junior College), Ong Xing Cong, Ernest Chong, Joel Tay (Raffles Junior College), Kenneth Tay (Anglo-Chinese Junior College) who gave similar solutions.

Consider each lock in turn. If there are less than 4 dwarfs with keys to the lock, then there exist 4 dwarfs who do not have the key to the lock which gives a contradiction. Thus there are at least 4 dwarfs with the key to the lock. Hence there is at most one choice of 3 dwarfs with no key to the lock. But there are $\binom{7}{3} = 35$ choices of 3 dwarfs but only 30 locks so there is at least one choice of 3 dwarfs who have keys to all the locks.