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Congruent Triangles – If You Look Carefully

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Congruent Triangles – If You Look Carefully

In this article we present a series of problems with a common theme. That theme is the exploitation of the properties of congruent triangles to solve a variety of problems which at first glance do not seem to involve congruence at all. However, once the would-be solver produces or recognizes congruent triangles in each problem, he is then able to move straight ahead to a solution.

Problem 1. In $\triangle ADC$, $\overline{DB}$ is perpendicular to $\overline{AC}$ at B so that $AB = 2$ and $BC = 3$ as shown in Figure 1.1. Furthermore, $\angle ADC = 45°$. Use this information to find the area of $\triangle ADC$.

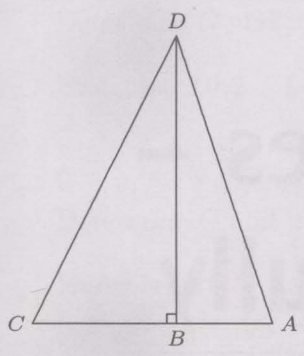

Figure 1.1 $\triangle ADC$.

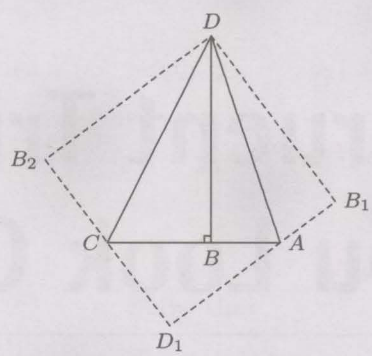

Figure 1.2 $\triangle ADC$ with congruent triangles.

Solution. Let us construct right $\triangle ADB_1$ with hypotenuse $\overline{AD}$ so that $\triangle ADB_1$ is congruent to $\triangle ADB$ as shown in Figure 1.2. In like manner, we construct right $\triangle CDB_2$ with hypotenuse $\overline{CD}$ so that $\triangle CDB_2$ is congruent to $\triangle CDB$. We then extend $\overline{B_1A}$ and $\overline{B_2C}$ to meet at D_1.

Since $\triangle ADB_1 \cong \triangle ADB$ and $\triangle CDB_2 \cong \triangle CDB$, $\angle ADB_1 + \angle CDB_2 = \angle ADB + \angle CDB = \angle ADC = 45°$. Therefore, $\angle B_1DB_2 = 90°$.

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Since $\angle B_1 = \angle B_2 = 90°$ and the sum of the angles of quadrilateral $DB_1D_1B_2$ is $360°$, it follows that $\angle D_1 = 90°$ and that $DB_1D_1B_2$ is a rectangle.

Since $DB = DB_1 = DB_2$, the rectangle is also a square. Therefore, $DB = D_1B_1 = D_1B_2$ as well.

Now $AB_1 = AB = 2$ so that $D_1A = D_1B_1 - AB_1 = DB - 2$. Similarly, $D_1C = DB - 3$.

Since $\triangle AD_1C$ is a right triangle, $D_1A^2 + D_1C^2 = AC^2 = (AB + BC)^2 = 25$. Therefore

$$(DB-2)^2 + (DB-3)^2 = 25 \text{ or } 2DB^2 - 10DB - 12 = 0.$$

It follows that $DB = 6$.

The area of $\triangle ADC$ is $\frac{1}{2}DB \cdot AC = \frac{1}{2}(5)(6) = 15$. □

Problem 2. In the isosceles right triangle ABC of Figure 2.1, $\angle A = 90°$ and $AB = AC$. Suppose that D is the interior point of the triangle so that $\angle ABD = 30°$ and $AB = DB$. Prove that $AD = CD$.

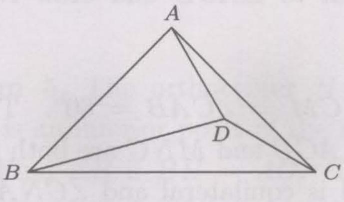

Figure 2.1 Isosceles Right Triangle ABC.

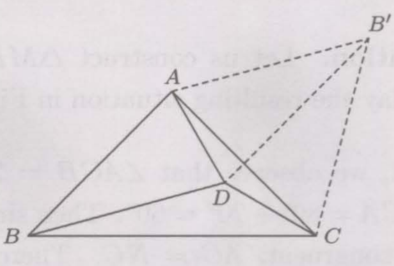

Figure 2.2 $\triangle ABC$ with $\triangle AB'D \cong \triangle ABD$.

Solution. Let us construct $\triangle AB'D$ to be congruent to $\triangle ABD$ and then draw $\overline{B'C}$ as suggested by Figure 2.2.

Since $AB = DB$, $\angle BAD = \angle BDA$. Then it follows that $\angle BAD = 75°$ since $\angle ABD = 30°$. Since $\triangle AB'D$ has been constructed congruent to $\triangle ABD$, $\angle B'AD = 75°$ as well. Then $\angle DAC = 90° - 75° = 15°$.

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We also have $\angle B'AC = \angle B'AD - \angle DAC = 60°$. Since it is given that $AB = AC$ and since $AB = AB'$ by construction, it follows that $AB' = AC$. Therefore $\triangle AB'C$ is equilateral.

Since $\angle AB'D = \angle ABD = 30°$, $\overline{B'D}$ must be the perpendicular bisector of $\overline{AC}$. Therefore, $AB'CD$ is a kite and $AD = CD$ as desired. □

Problem 3. Isosceles triangle ABC is shown in Figure 3.1. In that triangle, $\angle A = \angle B = 80°$ and cevian $\overline{AM}$ is drawn to side $\overline{BC}$ so that $CM = AB$. Find $\angle AMB$.

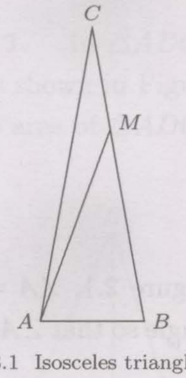

Figure 3.1 Isosceles triangle ABC.

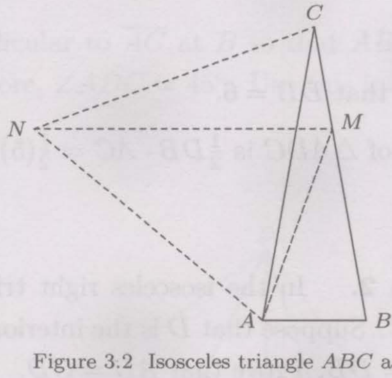

Figure 3.2 Isosceles triangle ABC and MNC.

Solution. Let us construct $\triangle MNC$ congruent to $\triangle ACB$ and draw $\overline{NA}$. We display the resulting situation in Figure 3.2.

Next, we observe that $\angle ACB = 20°$ and $\angle NCM = \angle CAB = 80°$. Therefore $\angle NCA = 80° - 20° = 60°$. Then since triangles ACB and MNC are both isosceles and congruent, $AC = NC$. Therefore $\triangle NCA$ is equilateral and $\angle CNA = 60°$. Thus $\angle ANM = 60° - 20° = 40°$.

We also see that $NA = NM$. Therefore $\angle NMA = \angle NAM = \frac{180°-40°}{2} = 70°$. In addition, $\angle AMC = \angle NMA + \angle NMC = 70° + 80° = 150°$. Thus

$$\angle AMB = 180° - \angle AMC = 180° - 150° = 30°.$$

□

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Problem 4. Triangle ABC is a right triangle with $\angle A = 30°$ and $\angle C = 90°$. Segment $\overline{DE}$ is perpendicular to $\overline{AC}$ at D and $AD = CB$ as indicated in Figure 4.1. Find DE if $DE + AC = 4$.

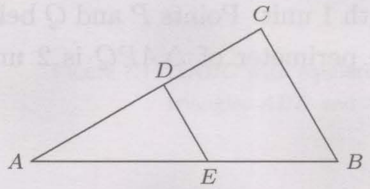

Figure 4.1 Right triangle ABC.

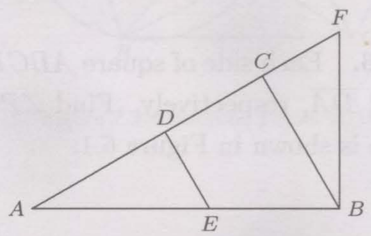

Figure 4.2 Right triangle ABC with $\triangle BCF \cong \triangle ADE$

Solution. Let us construct $\triangle BCF$ congruent to $\triangle ADE$. The result is as displayed in Figure 4.2 because $CB = AD$. Since $\angle BCF$ is a right angle, points A, D, C, and F are collinear.

Now $DE + AC = 4$ implies that $CF + AC = AF = 4$ since $CF = DE$ in congruent triangles BCF and ADE.

The congruence of the triangles also implies that $\angle CBF = \angle A = 30°$. Therefore $CF = BF \sin 30° = BF/2 = AF \sin 30°/2 = AF/4 = 4/4 = 1$.

We conclude that $DE = CF = 1$. □

Problem 5. The orthocenter H of $\triangle ABC$ is an interior point of the triangle. Find $\angle B$ if $BH = AC$. The feet of the altitudes from A and B are denoted by D and E, respectively. The geometry of the problem is shown in Figure 5.

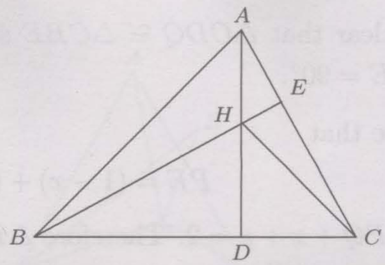

Figure 5. Triangle ABC with orthocenter H

Solution. Since $\overline{AD}$ and $\overline{BE}$ are altitudes of $\triangle ABC$, $\angle ADC = \angle BDH = 90°$. In right triangles ADC and BEC, $\angle CAD + \angle C = 90°$ and $\angle CBE + \angle C = \angle DBH +$

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$\angle C = 90°$, respectively. Thus $\angle CAD = \angle DBH$. Since it has been given that $BH = AC$, it follows that $\triangle ADC \cong \triangle BDH$.

Therefore, $AD = BD$ and $\triangle BDA$ is an isosceles right triangle. We see that $\angle ABC = 45°$.

□

Problem 6. Each side of square $ABCD$ has length 1 unit. Points P and Q belong to $\overline{AB}$ and $\overline{DA}$, respectively. Find $\angle PCQ$ if the perimeter of $\triangle APQ$ is 2 units. The square is shown in Figure 6.1.

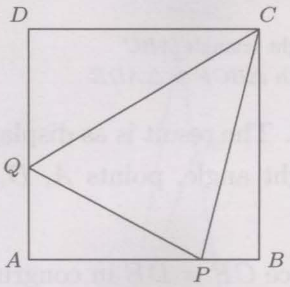

Figure 6.1 Square $ABCD$.

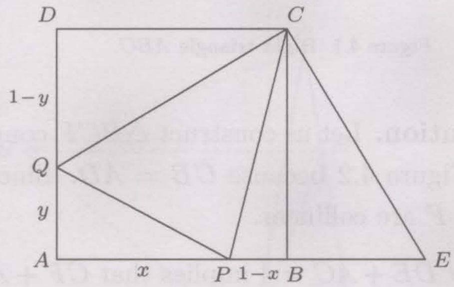

Figure 6.2 Square $ABCD$ with $\triangle CBE$.

Solution. Let us extend $\overline{AB}$ to E so that B is between A and E and $BE = DQ$. We also draw EC. We write that $AP = x$ and $AQ = y$. Then $PB = 1 - x$ and $DQ = BE = 1 - y$ as indicated on Figure 6.2.

It is clear that $\triangle CDQ \cong \triangle CBE$ so that $\angle DCQ = \angle BCE$. It is also clear that $\angle QCE = 90°$.

We see that

$$PE = (1 - x) + (1 - y) = 2 - (x + y) = PQ$$

since $PQ + x + y = 2$. Therefore $\triangle PCQ \cong \triangle PCE$. Therefore $\angle PCQ = \angle PCE = \frac{1}{2}\angle QCE = 45°$. □

Problem 7. We begin with $\triangle ABC$ and construct equilateral triangles ABD and ACE with their vertices D and E in the exterior of $\triangle ABC$. Segments $\overline{DC}$ and $\overline{EB}$ intersect at point P as shown in Figure 7.1. Find $\angle APD$.

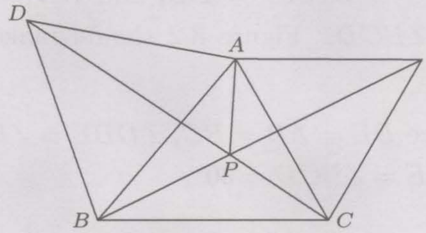

Figure 7.1 $\triangle ABC$ with equilateral triangles ABD and ACE.

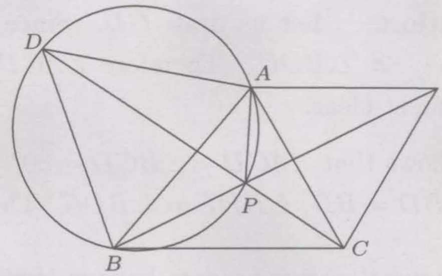

Figure 7.2 Concyclic Points A, D, B, P.

Solution. Since $\triangle ABD$ and $\triangle ACE$ are equilateral, $AD = AB$ and $AC = AE$. Also, $\angle DAC = 60° + \angle BAC = \angle BAE$. Therefore, $\triangle DAC \cong \triangle BAE$ by s.a.s. It follows that $\angle ADC = \angle ADP = \angle ABE = \angle ABP$.

Since segment $\overline{AP}$ subtends congruent angles $\angle ADP$ and $\angle ABP$, points A, D, B, and P are concyclic. Figure 7.2 shows the circle with chord $\overline{AP}$ and inscribed angles $\angle ADP$ and $\angle ABP$.

Since $\angle APD$ and $\angle ABD$ intercept the same arc, the angles have the same measure. Thus $\angle APD = \angle ABD = 60°$. □

Problem 8. Point D is an interior point of equilateral triangle ABC. It is given that $DA = DB$. Point E is also given so that $\angle DBE = \angle DBC$ and $BE = AB$. The geometry for this problem is displayed in Figure 8.1. Find $\angle E$.

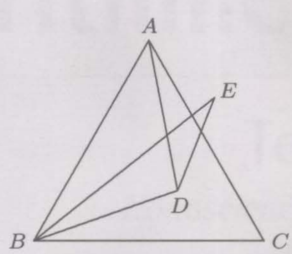

Figure 8.1 Equilateral $\triangle ABC$ and $\triangle BDE$.

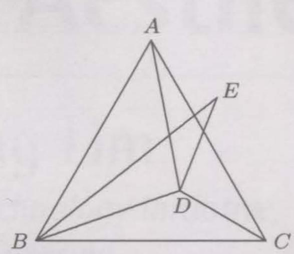

Figure 8.2 Equilateral $\triangle ABC$ and $\triangle BDE$ with CD added.

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Solution. Let us draw $\overline{CD}$. Since $CD = CD$, $AC = BC$, and $DA = DB$, $\triangle ADC \cong \triangle BDC$. Therefore $\angle ACD = \angle BCD$. Figure 8.2 should make this argument clear.

It follows that $\angle ACD = \angle BCD = 30°$. Since $BE = AB = BC$, $\angle DBE = \angle DBC$, and $BD = BD$, $\triangle BDE = \triangle BDC$. Thus $\angle E = \angle BCD = 30°$. □

We hope that our readers have enjoyed our problems and found them interesting exercises in geometrical reasoning.