Graphs and Their Applications (3)

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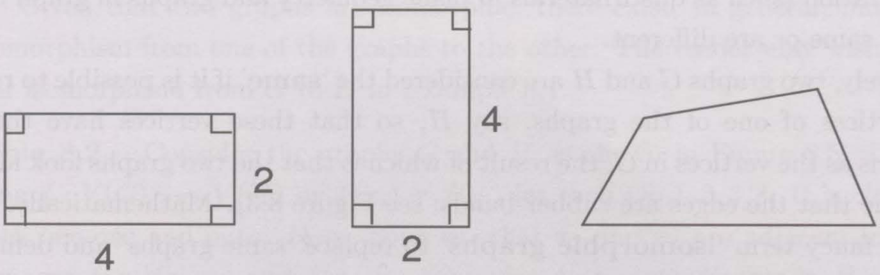
8. Isomorphic Graphs and Isomorphisms

Consider the following three quadrilaterals:



In **plane geometry**, we would say that the first two are the '**same**' (i.e., **congruent**), but they are '**different**' from the third one.

Let us now return to our **graph theory**. Suppose we are asked to draw a graph G which is defined as follows: its vertex set $V(G) = \{w, x, y, z\}$ and edge set $E(G) = \{wx, xy, yz, zw\}$. Some of us may place the four vertices as shown in Figure 8.1(a), others may place them as shown in Figure 8.1(b), (c), etc.

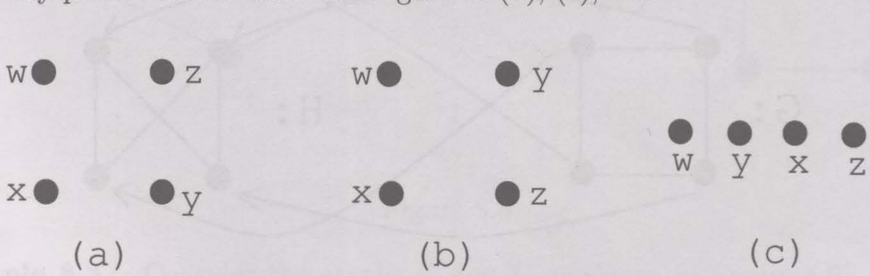


Figure 8.1

By joining some four pairs of vertices with the four edges as given in $E(G)$, we would have their corresponding diagrams as shown in Figure 8.2.

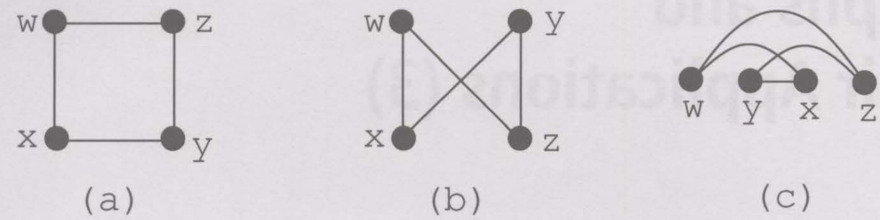


Figure 8.2

Apparently, these three diagrams look very different 'geometrically'. However, in the context of '**graphs**', they are absolutely the **same**.

In any area of mathematics (such as plane geometry and graph theory), the first thing to do before proceeding any further is to know whether two objects under consideration (such as quadrilaterals in plane geometry and graphs in graph theory) are the same or are different.

Intuitively, two graphs G and H are considered the '**same**' if it is possible to relocate the vertices of one of the graphs, say H, so that these vertices have the same positions as the vertices in G, the result of which is that the two graphs look identical (imagine that the edges are rubber bands; see Figure 8.3). Mathematically, we use a more fancy term '**isomorphic graphs**' to replace 'same graphs' and define that the graphs G and H are **isomorphic** if there exists a one-one and onto mapping $f : V(G) \longrightarrow V(H)$ such that two vertices u, v are adjacent in G when and only when their images $f(u)$ and $f(v)$ under f are adjacent in H (i.e., **the adjacency is preserved under** f). In this case, we shall write $G \cong H$ and call the mapping f an **isomorphism** from G to H. (The word 'isomorphism' is derived from the Greek words *isos*(meaning 'equal') and *morphe*(meaning 'form').)

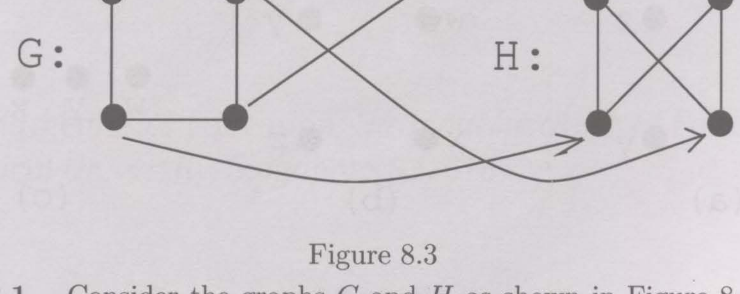


Figure 8.3

Example 8.1. Consider the graphs G and H as shown in Figure 8.4. We claim that $G \cong H$. Indeed, if we define a mapping $f : V(G) \longrightarrow V(H)$ by $f(u_i) = v_i$ for each $i = 1, 2, \cdots, 6$, then it can be checked that f is both one-one and onto, and that adjacency is preserved under f. Thus G and H are isomorphic under the isomorphism f.

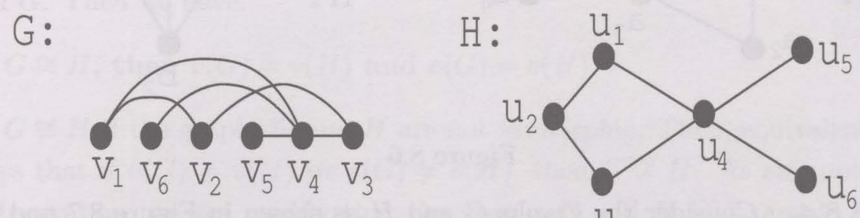


Figure 8.4

Note. Given that two graphs are isomorphic, there exist, in general, more than one isomorphism from one of the graphs to the other. The reader may wish to find another isomorphism from G to H in Example 8.1.

Example 8.2. Consider the graphs G and H as shown in Figure 8.5. Define a mapping $f : V(G) \longrightarrow V(H)$ by $f(x_i) = f(y_i)$ for each $i = 1, 2, 3, 4$. It is clear that f is both one-one and onto. Note, however, that x_2 and x_4 are adjacent in G but their images $f(x_2)(= y_2)$ and $f(x_4)(= y_4)$ under f are not adjacent in H. Thus f does not preserve adjacency, and so f is **not** an isomorphism from G to H.

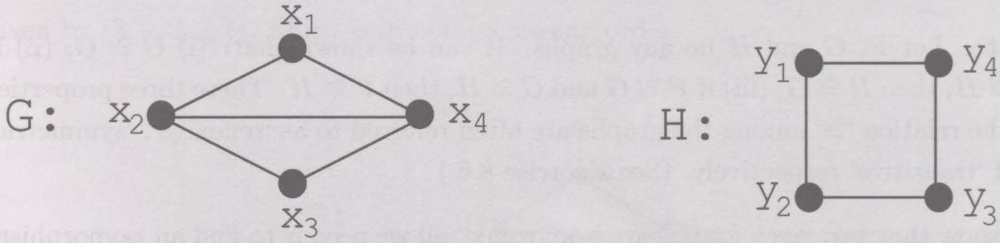


Figure 8.5

Example 8.3. Consider the graphs G and H as shown in Figure 8.6. Define a mapping $f : V(G) \longrightarrow V(H)$ by $f(a_i) = f(b_i)$ for each $i = 1, 2, 3, 4$. It is obvious that f is both one-one and onto. Observe, however, that a_2 and a_4 are not adjacent in G but their images $f(a_2)(= b_2)$ and $f(a_4)(= b_4)$ are adjacent in H. Thus f does not preserve adjacency, and so f is **not** an isomorphism from G to H.

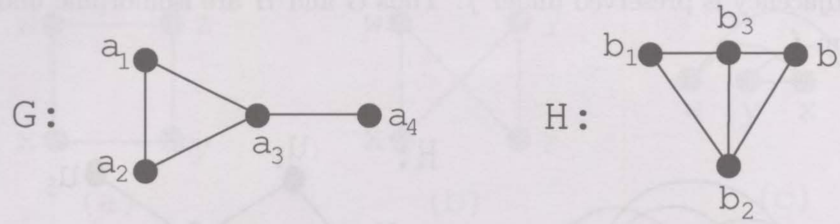


Figure 8.6

Example 8.4. Consider the graphs G and H as shown in Figure 8.7 and define a mapping $f : V(G) \longrightarrow V(H)$ by $f(w_i) = f(z_i)$ for each $i = 1, 2, \cdots, 5$. Though f is both one-one and onto, it is clear that f does not preserve adjacency, and so f is not an isomorphism from G to H. However, it does not mean that G is not isomorphic to H. Indeed, $G \cong H$ and the mapping $g : V(G) \longrightarrow V(H)$, defined by $g(w_1) = z_4$, $g(w_2) = z_2$, $g(w_3) = z_5$, $g(w_4) = z_3$ and $g(w_5) = z_1$, is an isomorphism from G to H.

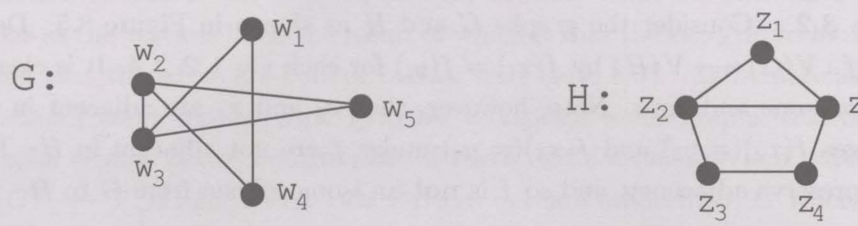


Figure 8.7

Note. Let F, G and H be any graphs. It can be shown that: (i) $G \cong G$; (ii) if $G \cong H$, then $H \cong G$; (iii) if $F \cong G$ and $G \cong H$, then $F \cong H$. These three properties of the relation '$\cong$' among the graphs are often referred to as 'reflexive', 'symmetric' and 'transitive' respectively. (See Exercise 8.6.)

To show that two given graphs are isomorphic, all we need is to find an isomorphism between them as done in Examples 8.1 and 8.4. How about showing that two given graphs are not isomorphic? Can we simply say that it is 'so' because there is no isomorphism between them? This argument is certainly not convincing in general unless we do list all the possible one-one and onto mappings between the two vertex sets (which are, however, too many if the numbers of the vertices of the graphs considered are large) for checking.

Recall that two graphs are isomorphic if we can find a one-one and onto mapping between their vertex sets which preserves adjacency. It follows readily that if $G \cong$

H, then they must have the same number of vertices and same number of edges respectively. For convenience, let us now denote by $v(G)$ the number of vertices in G and call it the **order** of G; and by $e(G)$ the number of edges in G and call it the **size** of G. Then we have:

(1) **If** $G \cong H$**, then** $v(G) = v(H)$ **and** $e(G) = e(H)$**.**

Write $G \not\cong H$ if the graphs G and H are not isomorphic. Then, equivalently, result (1) says that if $v(G) \neq v(H)$ **or** $e(G) \neq e(H)$, then $G \not\cong H$. As an application of this observation, we see readily that the graphs G and H in Example 8.2 (resp., Example 8.3) are not isomorphic, and that no two of the four graphs given in Figure 8.8 can be isomorphic.

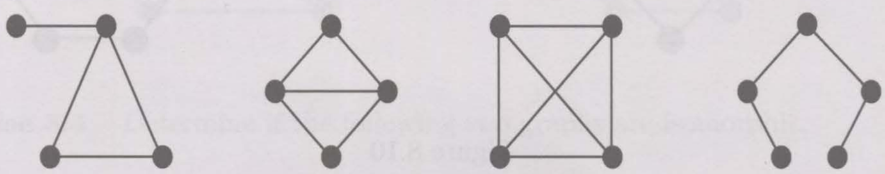


Figure 8.8

Recall that the degree of a vertex v in a graph G, denoted by $d(v)$, is the number of edges incident with it. Assume that $V(G) = \{u_1, u_2, \cdots, u_n\}$. Call the sequence $(d(u_1), d(u_2), \cdots, d(u_n))$ the **degree sequence** of G. We may rename the vertices in G so that $d(u_1) \geq d(u_2) \geq \cdots \geq d(u_n)$. For instance, in the graph G of Figure 8.9, the five vertices are named as $u_1, \cdots, u_5$ so that the degree sequence of G is given by $(3, 2, 2, 2, 1)$, which is in non-increasing order.

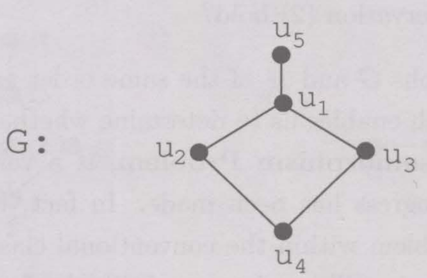


Figure 8.9

Suppose that two graphs G and H are isomorphic under an isomorphism f. As f preserves adjacency, it follows that, for each vertex v in G, $d(v) = d(f(v))$ (see Exercise 8.7). Thus, we have:

(2) **If $G \cong H$, then G and H have the same degree sequence, in non-increasing order.**

Hence, equivalently, if G and H have different degree sequences in non-increasing order, then $G \ncong H$. As an application of this observation, let us consider the following:

Example 8.5. Determine whether the graphs of Figure 8.10 are isomorphic:

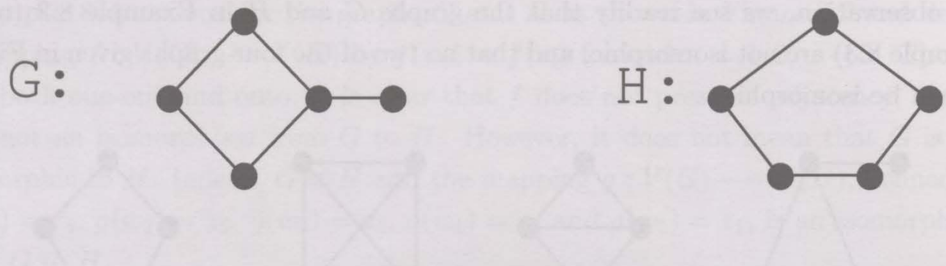


Figure 8.10

The degree sequence of G is $(3, 2, 2, 2, 1)$ while that of H is $(2, 2, 2, 2, 2)$, which are different. Thus, $G \ncong H$.

Remarks. (i) Actually, in Example 8.5, we don't need to use that 'big' notion of the degree sequence to conclude that $G \ncong H$. We could arrive at the same by simply pointing out a simple fact that G has an end-vertex (a vertex of degree 1) while H does not have.

(ii) The graphs G and H in Example 8.5 are of the same order and size respectively, yet they are not isomorphic. This shows that the converse of observation (1) is false. Does the converse of observation (2) hold?

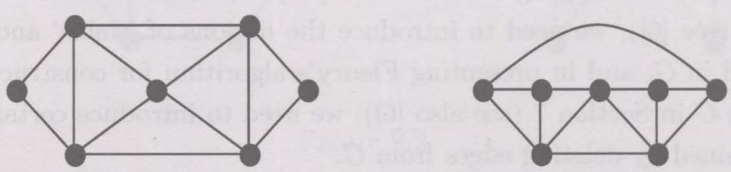
Given two arbitrary graphs G and H of the same order and same size, is there an 'efficient' procedure which enables us to determine whether $G \cong H$? This problem, known as the **Graph Isomorphism Problem**, is a very difficult problem, and until now, only little progress has been made. In fact, the exact location of the Graph Isomorphism Problem within the conventional classifications of algorithmic (procedural) complexities is still not known. Still, the fact is that there are many practical applications which desire a fast procedure to test graph isomorphism. For example, organic chemists who routinely deal with graphs which represent molecular links would like some system to quickly give each graph a unique name. Thus, many research papers have been published which discuss how to build fast and practical isomorphism testers.

For a good survey on the Graph Isomorphism Problem, the reader may refer to the paper by Fortin [2] or the book by Kobler et al [4].

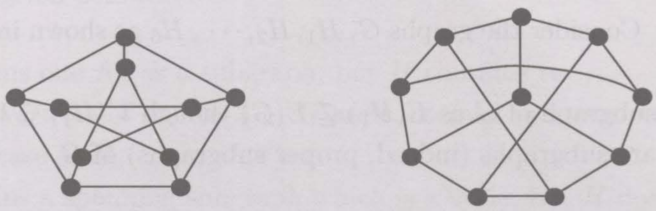
Exercise 8.1. Draw all non-isomorphic graphs of order n with $1 \leq n \leq 4$.

Exercise 8.2. (i) Draw all non-isomorphic graphs of order 5 and size 3.
(ii) Draw all non-isomorphic graphs of order 5 and size 7.

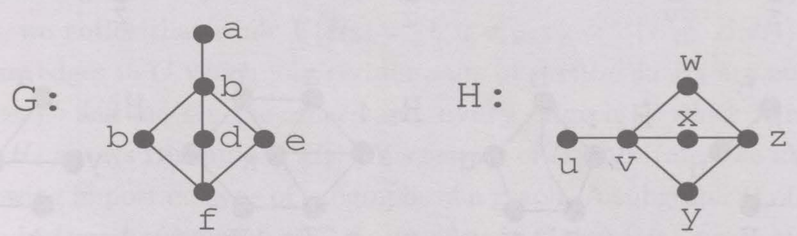
Exercise 8.3. Determine if the following two graphs are isomorphic.



Exercise 8.4. Determine if the following two graphs are isomorphic.



Exercise 8.5. The following two graphs G and H are isomorphic. List all the isomorphisms from G to H.



Exercise 8.6. Prove, by definition of an isomorphism, that the relation '$\cong$' is reflexive, symmetric and transitive among the family of graphs.

Exercise 8.7. Let f be an isomorphism from a graph G to a graph H and w a vertex in G. Show that the degree of w in G is equal to the degree of $f(w)$ in H.

9. Subgraphs of a Graph

In studying problems on a graph, quite often, we may wish to consider the 'graphical structures' of **certain portions** of the graph. For instance, in determining whether the graphs G and H in Example 8.2 are isomorphic, we may try the following way: observe that the vertices x_1, x_2 and x_4 in G form a K_3 (i.e., a triangle or a complete graph of order 3, see Section 3 in [5]), but there is no K_3 contained in H; this implies that G and H have different 'graphical structures', and so $G \not\cong H$. As another example, in studying the unicursal property of a connected multigraph G in Section 6 (see [6]), we need to introduce the notions of 'walks' and 'trails' that are contained in G, and in presenting Fleury's algorithm for constructing a closed Euler trail in G in Section 7 (see also [6]), we need to introduce certain parts of G that are obtained by deleting edges from G.

Let G be a graph. A graph H is called a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. By definition, every graph is a subgraph of itself. A subgraph H of G is said to be **proper** if $H \not\cong G$.

Example 9.1. Consider the graphs $G, H_1, H_2, \cdots, H_6$ as shown in Figure 9.1. We observe that

(1) H_1 is not a subgraph of G as $E(H_1) \not\subseteq E(G)$ though $V(H_1) \subseteq V(G)$;

(2) $H_2, \cdots, H_6$ are subgraphs (indeed, proper subgraphs) of G.

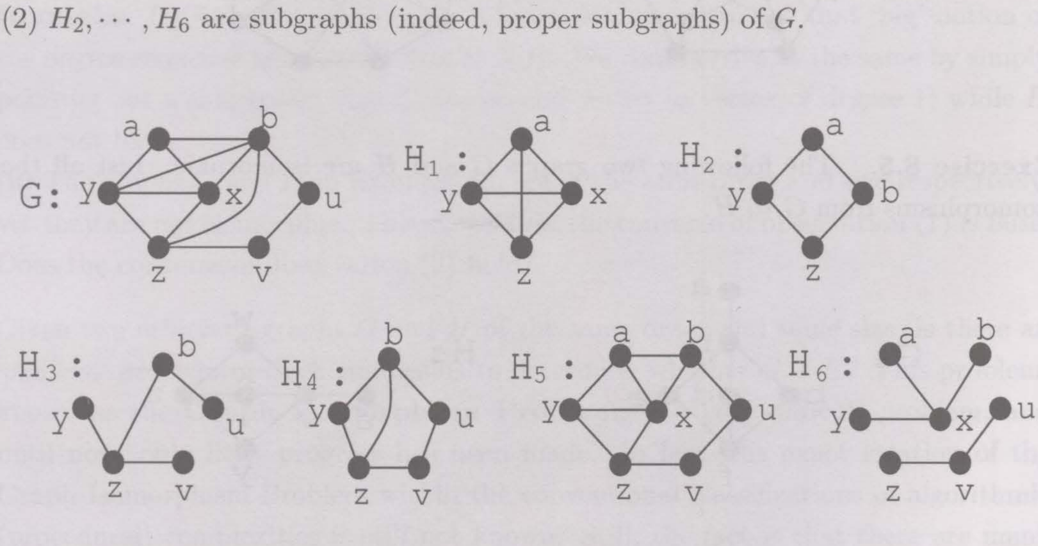


Figure 9.1

Note that $V(H_i) \neq V(G)$ for $i = 2, 3, 4$, but $V(H_5) = V(H_6) = V(G)$. In general, a subgraph H of a graph G is said to be **spanning** if $V(H) = V(G)$. Thus, in Example 9.1, the graphs H_5 and H_6 are spanning subgraphs of G, but H_2, H_3 and H_4 are not. It is easy to show (see Exercise 9.3) that **a subgraph H of a graph G is a**

spanning subgraph of G **if and only if** H **is obtained from** G **by deleting some edges in** G. For instance, in Example 9.1, we have $H_5 = G - \{by, bz\}$ and $H_6 = G - \{ay, ab, by, bz, bu, xz, yz\}$ (see Section 7 in [6] for notation).

Example 9.2. Consider the graphs G and H as shown in Figure 9.2.

Figure 9.2

Note that G and H have the same degree sequence in non-increasing order, that is, $(3, 3, 2, 2, 2, 2)$, yet $G \not\cong H$ (this shows that the converse of observation (2) in Section 8 is false). How do we argue that $G \not\cong H$? Some ways using the concept of 'subgraphs' are given below:

(i) G contains one K_3 as a subgraph, but H contains two;

(ii) the two vertices of degree 3 in G are contained in a common K_3, but this not the case in H;

(iii) G contains a spanning subgraph which is a cycle, but H does not have one;

(iv) G contains a cycle C_5 of order 5, but H does not have; etc.

Any one of the reasons above would be good enough to justify that $G \not\cong H$.

Look at the subgraphs H_3 and H_4 of G in Example 9.1. By comparing these two subgraphs, we notice that while $V(H_3) = \{b, u, v, y, z\} = V(H_4)$, $E(H_3) \neq E(H_4)$. In H_3, some edges in G which join certain pairs of vertices in H_3 are no longer in; for instance, yb and uv. On the other hand, **every edge** in G which joins a pair of vertices in H_4 always **remains** in H_4. This feature of H_4 motivates the introduction of the following important type of subgraphs of a graph. A subgraph H of a graph G is called an **induced subgraph** of G if **any edge** in G that joins a pair of vertices in H is also in H. If H is an induced subgraph of G, we also say that H is the subgraph **induced by its vertex set** $V(H)$ and we write $H = [V(H)]$. Thus, in Example 9.1, among the subgraphs $H_2, \cdots, H_6$ of G, only H_4 is an induced subgraph of G, and we see that H_4 is induced by $\{b, u, v, y, z\}$ (in notation, $H_4 = [\{b, u, v, y, z\}]$). The subgraphs of G in Example 9.1 induced by $\{a, x, y, z\}$ and $\{a, b, u, x, z\}$ are shown in (a) and (b) of Figure 9.3 respectively.

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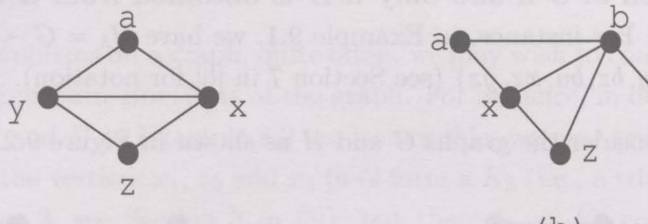


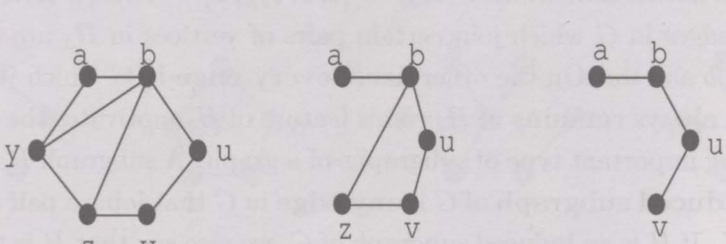
(a) (b)

Figure 9.3

A subgraph H of a graph G is not necessarily an induced subgraph of G. However, it can always be extended to an induced subgraph of G induced by $V(H)$ by adding to H all the missing edges existing in G.

We have seen that the spanning subgraphs of a graph G are those subgraphs of G that can be obtained from G by deleting some edges in G. In contrast with this, we shall see that induced subgraphs of G can be obtained from G as well, but by deleting some vertices in G as defined below.

Let G be a graph and W a set of vertices in G. We shall denote by $G - W$ the subgraph of G obtained by removing each vertex in W from $V(G)$ together with **all the edges incident with it** from $E(G)$. When W is a singleton, say $W = \{w\}$, we shall write $G - w$ for $G - \{w\}$. For instance, if G is the graph given in Example 9.1, then the subgraphs $G - x$, $G - \{x, y\}$ and $G - \{x, y, z\}$ of G are shown in (a) , (b) and (c) of Figure 9.4 respectively. Note that $G - \{x, y, z\} = [\{a, b, u, v\}]$, $G - \{x, y\} = [\{a, b, u, v, z\}]$ and $G - x = [\{a, b, u, v, y, z\}]$. In general, one can show (see Exercises 9.4 and 9.5) that a **subgraph W of a graph G is an induced subgraph of G if and only if $W = G - (V(G)\backslash V(W))$, where $V(G)\backslash V(W)$ consists of those vertices of G which are *not* in W.**



(a) (b) (c)

Figure 9.4

There is an edge version for induced subgraphs of a graph. Let G be a graph and F a set of edges in G. The subgraph of G induced by F, denoted by $[F]$, is the graph whose vertex set consists of those vertices incident with an edge in F and whose edge set is just F. For instance, if G is the graph given in Example 9.1, and $F_1 = \{ab, yz\}$ and $F_2 = \{ab, bx, xy, ya\}$, then the subgraphs $[F_1]$ and $[F_2]$ of G are shown in (a) and (b) of Figure 9.5 respectively.

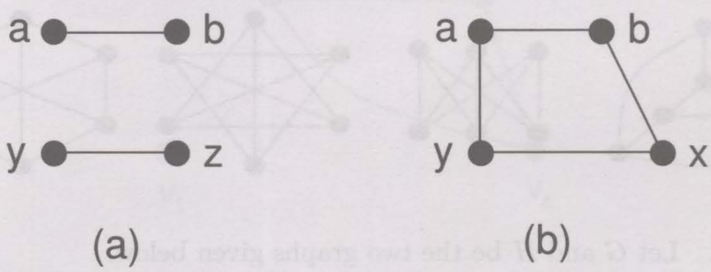
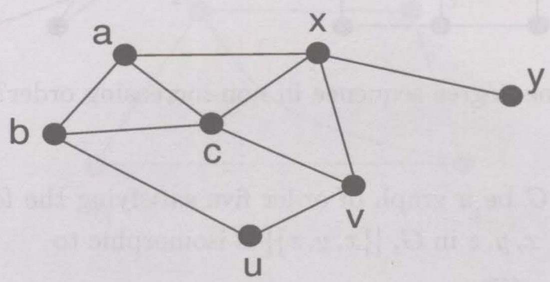


Figure 9.5

Exercise 9.1. Let G be the graph given below:



(i) Draw the subgraphs $[\{a, b, c, v, x\}]$, $[\{a, b, u, v, x\}]$ and $[\{ac, bc, cv, cx\}]$ of G.

(ii) Draw the subgraphs $G - \{ab, cv, xy\}$, $G - c$ and $G - \{b, v\}$ of G.

(iii) Draw the subgraphs $[V([\{ab, ac, vx\}])]$ and $G - V([\{ab, ac, vx\}])$ of G.

(iv) Draw the subgraph $G - E([\{a, b, c, x\}])$ of G.

(v) Draw a spanning subgraph of G that is connected and that contains a unique C_3 (a cycle of order 3) as a subgraph.

(vi) Draw a spanning subgraph of G that is connected and that contains no cycle as a subgraph.

Exercise 9.2. Let H be a spanning and induced subgraph of a graph G. What can be said of H?

Exercise 9.3. Let H be a subgraph of a graph G. Show that H is a spanning subgraph of G if and only if $H = G - F$, where $F \subseteq E(G)$.

Exercise 9.4. Let G be a graph and $X \subseteq V(G)$. Show that $G - X = [V(G) \backslash X]$.

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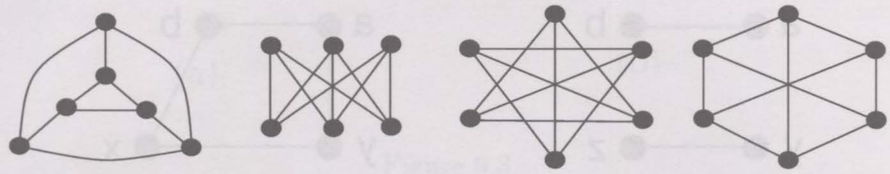
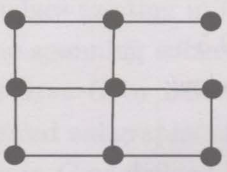
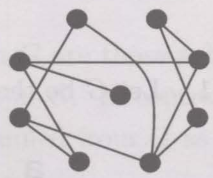
Exercise 9.5. Let G be a graph and W a subgraph of G. Show that W is an induced subgraph of G if and only if $W = G - (V(G)\backslash V(W))$.

Exercise 9.6. Determine which of the following four graphs are isomorphic and which are not so.

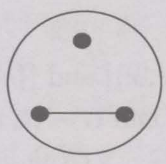
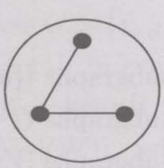


Exercise 9.7. Let G and H be the two graphs given below:

G:  H: 

Do they have the same degree sequence in non-increasing order? Are they isomorphic?

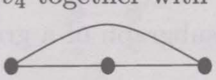
Exercise 9.8. Let G be a graph of order five satisfying the following condition: for any three vertices x, y, z in G, $[\{x, y, z\}]$ is isomorphic to

 or 

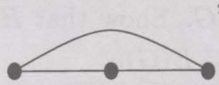
What is the graph G? Justify your answer.

10. The Reconstruction Conjecture

Let us begin with a simple problem. We are given a graph G with four vertices v_1, v_2, v_3 and v_4 together with the following information:

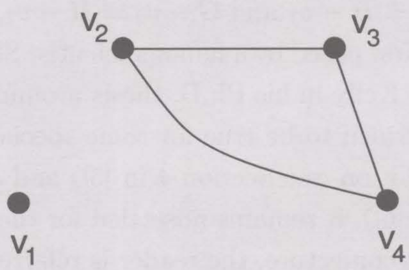
(i) $G - v_1 \cong$ ,

(ii) $G - v_2 \cong$,

(iii) $G - v_3 \cong$ ,

(iv) $G - v_4 \cong$ .

Our aim is to find out what G is.
By (i), G contains the following graph as a subgraph:



Now, by (iii), G must contain the graph of Figure 10.1 as a subgraph.

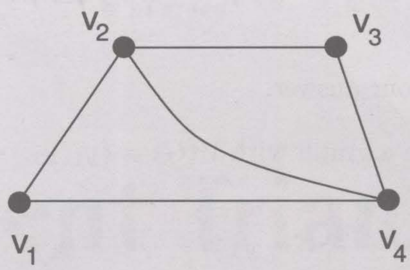


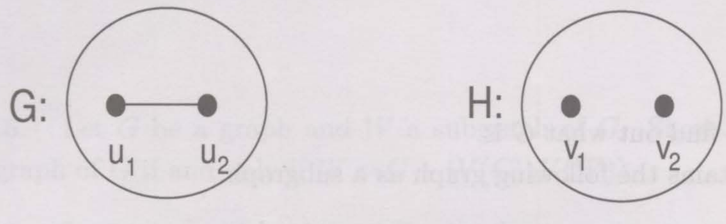
Figure 10.1

It is easily seen that this graph fulfills (ii) and (iv). Furthermore, it can be checked that this graph is the only graph that fulfills (i) to (iv). We thus conclude that G is the graph of Figure 10.1.

In general, let G be a graph with $V(G) = \{u_1, u_2, \cdots, u_n\}$. We say that G is **reconstructible** if, whenever H is a graph with $V(H) = \{v_1, v_2, \cdots, v_n\}$ such that $H - v_i = G - u_i$ for each $i = 1, 2, \cdots, n$, then $H \cong G$ (that is, G is uniquely determined by its n subgraphs: $G - u_1, G - u_2, \cdots, G - u_n$). Thus, the above example shows that the graph of Figure10.1 is reconstructible. A very well-known unsolved problem in graph theory is now stated below.

The Reconstuction Conjecture. Every graph of order at least three is reconstructible.

We note that a graph of order two is not reconstructible. Indeed, take G and H as shown below:



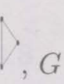
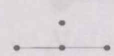
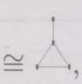
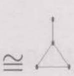
It is observed that $G - u_1 \cong H - v_1$ and $G - u_2 \cong H - v_2$, and yet $G \not\cong H$.
The above conjecture was first posed by a famous scientist SM Ulam (see also [7]) and was initially studied by PJ Kelly in his Ph.D. thesis around 1942 (see [3]) . Though the conjecture has been verified to be true for some special families of graphs such as regular graphs (for definition, see Section 4 in [5]) and disconnected graphs (for definition, see Section 5 in [6]), it remains unsettled for the general situation. For a very general survey on this conjecture, the reader is referred to the excellent article [1] given by Bondy.

Exercise 10.1. Let G be a graph with $V(G) = \{x_1, x_2, x_3, x_4\}$ such that

$G - x_1 \cong$ , $G - x_2 \cong$, $G - x_3 \cong$ and $G - x_4 \cong$
.

Determine G and justify your answer.

Exercise 10.2. Let G be a graph with $V(G) = \{y_1, y_2, \cdots, y_5\}$ such that

$G - y_1 \cong$ , $G - y_2 \cong$ , $G - y_3 \cong$ , $G - y_4 \cong$  and $G - y_5 \cong$ .
Determine G and justify your answer.

Exercise 10.3. Let G be a graph with $V(G) = \{u_1, u_2, \cdots, u_n\}$, where $n \geq 3$. Let $m = e(G)$, $m_i = e(G - u_i)$, $i = 1, 2, \cdots, n$. Show that
(i) $m = (m_1 + m_2 + \cdots + m_n)/(n - 2)$;
(ii) the degree of u_i in G is equal to $m - m_i$, $i = 1, 2, \cdots, n$.

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[5] K.M. Koh, Graphs and their applications (1), *Mathematical Medley* **29**(2) (2002), 86-94.

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