Number Pattern for Arithmetical Aesthetics

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Number Patterns for Arithmetical Aesthetics

The mathematician's patterns, like the painter's or the poet's, must be beautiful ....Beauty is the first test: there is no permanent place in the world for ugly mathematics.The best mathematics is serious as well as beautiful.... The 'seriousness' of a mathematical theorem lies, not in its practical consequence, which are usually negligible, but in the significance of the mathematical ideas which it connects [1].

Godfrey Harold Hardy (1877-1947)

1. Introduction

Why do some simply fall in love with mathematics while others cringe at the word? There are many reasons, and one reason is in the way mathematics is viewed: whether as a subject to be studied or as a means for training the mind, whether for examination purposes or for application purposes, whether to impress or to be impressed, whether it is something to get over with or something beautiful. One possible way of expressing the beauty of mathematics is through exposing its pattern, particularly in the arrangement of numbers. Hence number patterns refer to numbers arranged in such a manner that shows forth some form of repetition, be it ascending order, descending order or palindromic. Since these patterns contain some form of sequencing, they are known for their beauty and at times brought up as mathematical recreation. In this article, we shall see how number patterns can be expressed in arithmetic, and the conditions that allow these patterns to surface. The three categories of arithmetic covered are addition, multiplication and squaring. As much as possible, these number patterns are reflected both in the LHS and RHS of the arithmetical equations.

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2. Number Patterns in Arithmetic

Figure 1(a) shows summation of the form $(mmm\ldots)+(999\ldots)$ where $m = 1, 2, 3, \ldots 9$. A general pattern for these cases can be written as shown in Figure 1(b). It can be clearly seen that when $m = 2$, the result is a palindrome. Multiplications of the form $(mmm\ldots) \times (999\ldots)$ are furnished in Figure 2(a). The general pattern for multiplication shown in Figure 2(a) is depicted in Figure 2(b). Other cases of number pattern arising from multiplication involve $(333\ldots) \times (666\ldots)$ and $(222\ldots) \times (555\ldots)$, as shown in Figure 3. Again we note a palindromic pattern for the result of $(222\ldots) \times (555\ldots)$. Number patterns for squaring are given in Figure 4(a) for the cases of $(333\ldots)^2$, $(666\ldots)^2$ and $(999\ldots)^2$. These patterns can be expressed more generally as described in Figure 4(b). Other number patterns in squaring, such as $(333\ldots+m)^2$, $(666\ldots+m)^2$ and $(999\ldots+m)^2$ where $m = \pm 1$ and/or $m = \pm 2$, are displayed in Figures 5, 6 and 7 respectively.

3. Closure

Though number patterns may not find much application in technology and industry, their beauty cannot be denied. Number patterns, as objects of beauty, can present to the layman the aesthetic facet of mathematics, particularly that of numbers in a very lucid manner. And should number patterns spark the love of mathematics in the hearts of many a people, these patterns, then, are beautiful in more ways than one.

4. Exercise

Interested readers may want to work out these exercises to discover by themselves some interesting number patterns.

(a) Compare

$1^2 =$
$11^2 =$
$111^2 =$
$1111^2 =$

with $(222\ldots) \times (555\ldots)$ in Figure 3.

How are they related?

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(b) Compute the following

n	$f(n)$	Fill answer in this column
1	1	=
2	$1+2+1$	=
3	$1+2+3+2+1$	=
4	$1+2+3+4+3+2+1$	=
5	$1+2+3+4+5+4+3+2+1$	=

What is the relation between $f(n)$ and n?

(c) The following sequence has repetition of integer 9 on the LHS. Compute the integer sequence. (Hint: the answer is a palindrome, i.e. number pattern left to right is same as right to left).

n	$f(n)$	Fill answer in this column
1	$[9+2]^2$	=
2	$[99+2]^2$	=
3	$[999+2]^2$	=
4	$[9999+2]^2$	=
5	$[99999+2]^2$	=

Is there an easier way to describe $f(n)$?

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$$\begin{array}{rcl|rcl}
1+9 & = & 10 & 6+9 & = & 15 \\
11+99 & = & 110 & 66+99 & = & 165 \\
111+999 & = & 1110 & 666+999 & = & 1665 \\
1111+9999 & = & 11110 & 6666+9999 & = & 16665 \\
11111+99999 & = & 111110 & 66666+99999 & = & 166665 \\
 & & & & & \\
2+9 & = & 11 & 7+9 & = & 16 \\
22+99 & = & 121 & 77+99 & = & 176 \\
222+999 & = & 1221 & 777+999 & = & 1776 \\
2222+9999 & = & 12221 & 7777+9999 & = & 17776 \\
22222+99999 & = & 122221 & 77777+99999 & = & 177776 \\
 & & & & & \\
3+9 & = & 12 & 8+9 & = & 17 \\
33+99 & = & 132 & 88+99 & = & 187 \\
333+999 & = & 1332 & 888+999 & = & 1887 \\
3333+9999 & = & 13332 & 8888+9999 & = & 18887 \\
33333+99999 & = & 133332 & 88888+99999 & = & 188887 \\
 & & & & & \\
4+9 & = & 13 & 9+9 & = & 18 \\
44+99 & = & 143 & 99+99 & = & 198 \\
444+999 & = & 1443 & 999+999 & = & 1998 \\
4444+9999 & = & 14443 & 9999+9999 & = & 19998 \\
44444+99999 & = & 144443 & 99999+99999 & = & 199998 \\
 & & & & & \\
5+9 & = & 14 & & & \\
55+99 & = & 154 & & & \\
555+999 & = & 1554 & & & \\
5555+9999 & = & 15554 & & & \\
55555+99999 & = & 155554 & & &
\end{array}$$

(a)

$$\underbrace{\boxed{m}\boxed{\;\ldots\ldots\;}}_{\substack{n \text{ digits of} \\ \text{integer } m}} + \underbrace{\boxed{9}\boxed{\;\ldots\ldots\;}}_{\substack{n \text{ digits of} \\ \text{integer } 9}} = \boxed{1}\underbrace{\boxed{m}\boxed{\;\ldots\ldots\;}}_{\substack{(n-1) \text{ digits of} \\ \text{integer } m}}\boxed{m-1}$$

(b)

Figure 1. Number patterns of (a) $(mmm\ldots)+(999\ldots)$ and (b) its generalized pattern for $m=1,2,3,\ldots 9$.

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1×9	$= 09$	6×9	$= 54$
11×99	$= \mathbf{1089}$	66×99	$= \mathbf{6534}$
111×999	$= \mathbf{110889}$	666×999	$= \mathbf{665334}$
1111×9999	$= \mathbf{11108889}$	6666×9999	$= \mathbf{66653334}$
11111×99999	$= \mathbf{1111088889}$	66666×99999	$= \mathbf{6666533334}$
2×9	$= 18$	7×9	$= 63$
22×99	$= \mathbf{2178}$	77×99	$= \mathbf{7623}$
222×999	$= \mathbf{221778}$	777×999	$= \mathbf{776223}$
2222×9999	$= \mathbf{22217778}$	7777×9999	$= \mathbf{77762223}$
22222×99999	$= \mathbf{2222177778}$	77777×99999	$= \mathbf{7777622223}$
3×9	$= 27$	8×9	$= 72$
33×99	$= \mathbf{3267}$	88×99	$= \mathbf{8712}$
333×999	$= \mathbf{332667}$	888×999	$= \mathbf{887112}$
3333×9999	$= \mathbf{33326667}$	8888×9999	$= \mathbf{88871112}$
33333×99999	$= \mathbf{3333266667}$	88888×99999	$= \mathbf{8888711112}$
4×9	$= 36$	9×9	$= 81$
44×99	$= \mathbf{4356}$	99×99	$= \mathbf{9801}$
444×999	$= \mathbf{443556}$	999×999	$= \mathbf{998001}$
4444×9999	$= \mathbf{44435556}$	9999×9999	$= \mathbf{99980001}$
44444×99999	$= \mathbf{4444355556}$	99999×99999	$= \mathbf{9999800001}$
5×9	$= 45$		
55×99	$= \mathbf{5445}$		
555×999	$= \mathbf{554445}$		
5555×9999	$= \mathbf{55544445}$		
55555×99999	$= \mathbf{5555444445}$		

(a)

$$\underbrace{\boxed{m}\boxed{\ldots\ldots}}_{n \text{ digits of integer } m} \times \underbrace{\boxed{9}\boxed{\ldots\ldots}}_{n \text{ digits of integer } 9} = \underbrace{\boxed{m}\boxed{\ldots\ldots}}_{(n-1) \text{ digits of integer } m}\boxed{m-1}\underbrace{\boxed{9-m}\boxed{\ldots\ldots}}_{(n-1) \text{ digits of integer } (9-m)}\boxed{10-m}$$

(b)

Figure 2. Number patterns of (a) $(mmm\ldots) \times (999\ldots)$ and (b) its generalized pattern for $m = 1, 2, 3, \ldots 9$.

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3×6	$=$	18	2×5	$=$	1	$\times$	10
33×66	$=$	**2178**	22×55	$=$	**121**	$\times$	10
333×666	$=$	**221778**	222×555	$=$	**12321**	$\times$	10
3333×6666	$=$	**22217778**	2222×5555	$=$	**1234321**	$\times$	10
33333×66666	$=$	**2222177778**	22222×55555	$=$	**123454321**	$\times$	10

Figure 3. Number patterns for $(333\ldots) \times (666\ldots)$ and $(222\ldots) \times (555\ldots)$.

3^2	$=$	09	9^2	$=$	81
33^2	$=$	**1089**	99^2	$=$	**9801**
333^2	$=$	**110889**	999^2	$=$	**998001**
3333^2	$=$	**11108889**	9999^2	$=$	**99980001**
33333^2	$=$	**1111088889**	99999^2	$=$	**9999800001**
6^2	$=$	36			
66^2	$=$	**4356**			
666^2	$=$	**443556**			
6666^2	$=$	**44435556**			
66666^2	$=$	**4444355556**			

(a)

$$\underbrace{\boxed{m \mid \ldots\ldots}^{\,2}}_{\substack{n \text{ digits of} \\ \text{integer } m=3,6,9}} = \underbrace{\boxed{(m/3)^2 \mid \ldots}}_{\substack{(n-1) \text{ digits of} \\ \text{integer } (m/3)^2}}\boxed{(m/3)^2-1}\underbrace{\boxed{9-(m/3)^2 \mid \ldots}}_{\substack{(n-1) \text{ digits of} \\ \text{integer } 9-(m/3)^2}}\boxed{10-(m/3)^2}$$

(b)

Figure 4. Number patterns for (a) the square of $(333\ldots)$, $(666\ldots)$ and $(999\ldots)$, and (b) its generalized pattern with $m = 3, 6, 9$.

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$$
\begin{array}{rcl|rcl}
2^2 & = & 04 & 5^2 & = & 25 \\
32^2 & = & 1024 & 35^2 & = & 1225 \\
332^2 & = & 110224 & 335^2 & = & 112225 \\
3332^2 & = & 11102224 & 3335^2 & = & 11122225 \\
33332^2 & = & 1111022224 & 33335^2 & = & 111122222 \\
 & & & & & \\
4^2 & = & 16 & & & \\
34^2 & = & 1156 & & & \\
334^2 & = & 111556 & & & \\
3334^2 & = & 11115556 & & & \\
33334^2 & = & 1111155556 & & &
\end{array}
$$

Figure 5. Number patterns for the square of $(333\ldots + m)$.

$$
\begin{array}{rcl|rcl}
5^2 & = & 25 & 8^2 & = & 64 \\
65^2 & = & 4225 & 68^2 & = & 4624 \\
665^2 & = & 442225 & 668^2 & = & 446224 \\
6665^2 & = & 44422225 & 6668^2 & = & 44462224 \\
66665^2 & = & 4444222225 & 66668^2 & = & 4444622224 \\
 & & & & & \\
7^2 & = & 49 & & & \\
67^2 & = & 4489 & & & \\
667^2 & = & 444889 & & & \\
6667^2 & = & 44448889 & & & \\
66667^2 & = & 4444488889 & & &
\end{array}
$$

Figure 6. Number patterns for the square of $(666\ldots + m)$.

$$
\begin{array}{rcl|rcl}
7^2 & = & 49 & 8^2 & = & 64 \\
97^2 & = & 9409 & 98^2 & = & 9604 \\
997^2 & = & 994009 & 998^2 & = & 996004 \\
9997^2 & = & 99940009 & 9998^2 & = & 99960004 \\
99997^2 & = & 9999400009 & 99998^2 & = & 9999600004
\end{array}
$$

Figure 7. Number patterns for the square of $(999\ldots + m)$.

5. Answers to Exercise Problems

(a) $(222\ldots) \times (555\ldots) = 2 \sum_{i=0}^{n} 10^i \times 5 \sum_{i=0}^{n} 10^i = 10 \left(\sum_{i=0}^{n} 10^i \right)^2 \equiv 10 \times (111\ldots)^2.$

(b) $f(n) \equiv \left(\sum_{i=1}^{n-1} i \right) + n + \sum_{i=n-1}^{1} i = n + \sum_{i=1}^{n-1} [i + (n-i)] = n^2.$

(c) Yes. Since $9 \sum_{i=1}^{n} 10^{i-1} = 10^n - 1$, then

$$\left(9 \sum_{i=1}^{n} 10^{i-1} + 2 \right)^2 \equiv f(n) = (10^n + 1)^2 = 10^{2n} + 2(10)^n + 1.$$

References

[1] James R. Newman, ed., *The World of Mathematics*, pp.2027-2029.