The Power of Series

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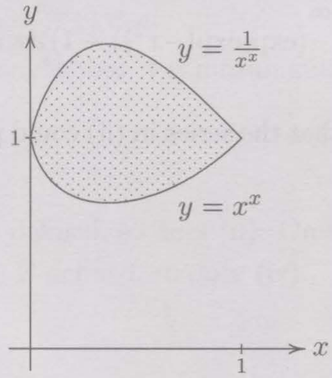
The use of series cannot be undermined. I was exploring what could be a nice way to evaluate or estimate the area trapped between the graphs of $y = x^x$ and $y = x^{-x}$ in (0,1], without the use of software. In this article, I proved that the desired area could be represented by the series $2\sum_{k=1}^{\infty} \frac{1}{(2k)^k}$.



Note that this series converges very rapidly, from just the first 4 terms, we can obtain a very good estimate of the area, which is about 0.507855486. Let us start by looking at the two curves. The area enclosed is

$$\int_0^1 \left(x^{-x} - x^x\right) dx = \int_0^1 x^{-x}\, dx - \int_0^1 x^x\, dx.$$

Since $e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!}$ for $-\infty < u < \infty$, therefore $e^{-x\ln x} = \sum_{k=0}^{\infty} \frac{(-x\ln x)^k}{k!}$ for $x > 0$.

Hence, $\int_0^1 x^{-x}\, dx = \int_0^1 e^{-x\ln x}\, dx = \int_0^1 \sum_{k=0}^{\infty} \frac{(-x\ln x)^k}{k!}\, dx.$

By Uniform Convergence Theorem, we can interchange the integral and summation operations. Thus,

$$\int_0^1 x^{-x}\, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_0^1 (x\ln x)^k\, dx.$$

Integral Calculus tells us that

$$\int_0^1 (x\ln x)^k\, dx = \frac{(-1)^k k!}{(k+1)^{k+1}} \qquad \text{for } k = 0, 1, 2, \ldots$$

Hence,

$$\int_0^1 x^{-x}\, dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(-1)^k k!}{(k+1)^{k+1}} = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{k+1}} = \sum_{k=1}^{\infty} \frac{1}{k^k}.$$

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Similarly, we can conclude that $\int_0^1 x^x\,dx = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^k}$. Therefore, the desired area is

$$\sum_{k=1}^{\infty} \frac{1}{k^k} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^k} = \sum_{k=0}^{\infty} \frac{1-(-1)^{k+1}}{k^k} = 2\sum_{k=1}^{\infty} \frac{1}{(2k)^{2k}}.$$

I would like to end by posing two problems to the readers. Prove the following.

(i) $\int_0^{\infty} (\exp(\exp(-x^{\alpha})) - 1)\,dx = \frac{1}{\alpha}\Gamma(\frac{1}{\alpha}) \sum_{k=0}^{\infty} \frac{1}{k!\,\sqrt[\alpha]{k}}$, for $\alpha > 0$.

(ii) $\int_0^{\infty} (\exp(\exp(-x^2)) - 1)\,dx = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{1}{k!\sqrt{k}}$.

Note that the series in (ii) could probably lead us to explore some interesting aspects of π.