APPLICATION OF PASCAL'S TRIANGLE IN REPRESENTING FINITE DIFFERENCE SOLUTION TO PARTIAL DIFFERENTIAL EQUATION

TEIK-CHENG LIM

1. Motivation

In this article, the reader will observe a certain number pattern existing in the finite difference (FD) solution to a class of partial differential equations (PDE). This number pattern is none other than the Pascal's Triangle. Hence by constructing the Pascal's Triangle, one can save time and easily obtain the FD equation corresponding to the PDE. Since every element (r) from every row (n) of the Pascal's Triangle can be described by the binomial coefficient, one can attempt some form of generalization to the FD solution of the PDE.

But first, let us go back in time and be motivated by the historical development of Pascal's Triangle. We shall call this Pascal's Triangle as simply "The Triangle", for it was discovered long before the birth of Blaise Pascal, whose name we adopt for the "Triangle" at the present time. The modern representation of this "Triangle" is depicted in Figure 1(a). The "Triangle" may have been known in ancient India during the time of Halayudha, who may lived in the 900s AD [1], although the original manuscript depicting the "Triangle" may not be well known. Possibly the oldest manuscript showing the "Triangle" is found in the works of Omar Khayyam, a great astronomer, poet and mathematician in Persia (now Iran) during 1000s AD, in the context of extracting roots of numbers. See Figure 1(b) for Khayyam's Triangle. In the Far East, the "Triangle" was published in a Chinese book entitled "Precious Mirror of the Four Elements" by Shih-Chieh Chu (Shijie Zhu) in the year 1303 AD [2]. See Chu's Triangle in Figure 1(c). Even then, the triangle was considered ancient in Chu's time as it had been discovered by Chia Hsien in obtaining coefficients of binomial expansion some time in 1000s AD [3]. In Europe, the "Triangle" came to be known as Pascal's Triangle with Blaise Pascal's completion of "Traite du Triangle Arithmetique" in 1654 (see [4] for recent edition). Pascal's version of the "Triangle" is shown in Figure 1(d), and was motivated by the need to find out how many ways are there to select r number of objects from n number of objects. In addition to the above, Pascal's Triangle is well known for obtaining:

Application of Pascal's Triangle

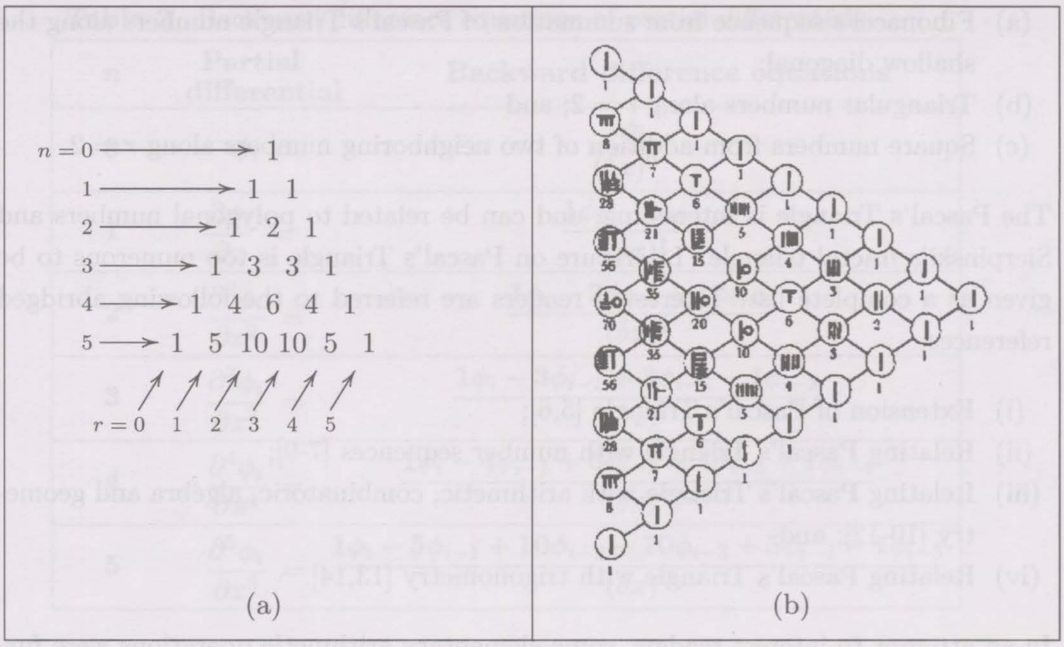

Figure 1. The "Triangle" written in the (a) modern form, (b) Omar Khayyam's form, (c) Shih-Chieh Chu's form, and (d) Blaise Pascal's form.

Application of Pascal's Triangle

(a) Fibonacci's sequence from summation of Pascal's Triangle numbers along the shallow diagonal;
(b) Triangular numbers along $r = 2$; and
(c) Square numbers from addition of two neighboring numbers along $r = 2$.

The Pascal's Triangle is interesting, and can be related to polygonal numbers and Sierpinski's fractal triangle. Literature on Pascal's Triangle is too numerous to be given as a complete list. Interested readers are referred to the following abridged references:

(i) Extension of Pascal's Triangle [5,6];
(ii) Relating Pascal's Triangle with number sequences [7-9];
(iii) Relating Pascal's Triangle with arithmetic, combinatoric, algebra and geometry [10-12]; and
(iv) Relating Pascal's Triangle with trigonometry [13,14].

In an attempt to interest readers, some elementary arithmetic operations were furnished in order to display number patterns [15]. As an extension to the previous article [15], we explore the occurrence of Pascal's Triangle in the finite difference solution to a class of partial differentials.

2. Pascal's Triangle in Difference Equations

Observe now the forward difference and backward difference forms to the partial differential $\frac{\partial^b \phi_i}{\partial x^n}$ shown in Tables 1 and 2.

Table 1 Forward difference equations of partial differentials.

n	Partial differential	Forward difference equations
0	$\phi_i =$	$\dfrac{\mathbf{1}\phi_i}{(\delta x)^0}$
1	$\dfrac{\partial \phi_i}{\partial x} =$	$\dfrac{\mathbf{1}\phi_{i+1} - \mathbf{1}\phi_i}{(\delta x)^1}$
2	$\dfrac{\partial^2 \phi_i}{\partial x^2} =$	$\dfrac{\mathbf{1}\phi_{i+2} - \mathbf{2}\phi_{i+1} + \mathbf{1}\phi_i}{(\delta x)^2}$
3	$\dfrac{\partial^3 \phi_i}{\partial x^3} =$	$\dfrac{\mathbf{1}\phi_{i+3} - \mathbf{3}\phi_{i+2} + \mathbf{3}\phi_{i+1} - \mathbf{1}\phi_i}{(\delta x)^3}$
4	$\dfrac{\partial^4 \phi_i}{\partial x^4} =$	$\dfrac{\mathbf{1}\phi_{i+4} - \mathbf{4}\phi_{i+3} + \mathbf{6}\phi_{i+2} - \mathbf{4}\phi_{i+1} + \mathbf{1}\phi_i}{(\delta x)^4}$
5	$\dfrac{\partial^5 \phi_i}{\partial x^5} =$	$\dfrac{\mathbf{1}\phi_{i+5} - \mathbf{5}\phi_{i+4} + \mathbf{10}\phi_{i+3} - \mathbf{10}\phi_{i+2} + \mathbf{5}\phi_{i+1} - \mathbf{1}\phi_i}{(\delta x)^5}$

Application of Pascal's Triangle

Table 2 Backward difference equations of partial differentials.

n	**Partial differential**	**Backward difference equations**
0	$\phi_i =$	$\dfrac{\mathbf{1}\phi_i}{(\delta x)^0}$
1	$\dfrac{\partial \phi_i}{\partial x} =$	$\dfrac{\mathbf{1}\phi_i - \mathbf{1}\phi_{i-1}}{(\delta x)^1}$
2	$\dfrac{\partial^2 \phi_i}{\partial x^2} =$	$\dfrac{\mathbf{1}\phi_i - \mathbf{2}\phi_{i-1} + \mathbf{1}\phi_{i-2}}{(\delta x)^2}$
3	$\dfrac{\partial^3 \phi_i}{\partial x^3} =$	$\dfrac{\mathbf{1}\phi_i - \mathbf{3}\phi_{i-1} + \mathbf{3}\phi_{i-2} - \mathbf{1}\phi_{i-3}}{(\delta x)^3}$
4	$\dfrac{\partial^4 \phi_i}{\partial x^4} =$	$\dfrac{\mathbf{1}\phi_i - \mathbf{4}\phi_{i-1} + \mathbf{6}\phi_{i-2} - \mathbf{4}\phi_{i-3} + \mathbf{1}\phi_{i-4}}{(\delta x)^4}$
5	$\dfrac{\partial^5 \phi_i}{\partial x^5} =$	$\dfrac{\mathbf{1}\phi_i - \mathbf{5}\phi_{i-1} + \mathbf{10}\phi_{i-2} - \mathbf{10}\phi_{i-3} + \mathbf{5}\phi_{i-4} - \mathbf{1}\phi_{i-5}}{(\delta x)^5}$

Each of these equations of order n is obtained by taking difference from difference equations of order $(n-1)$. As such, deriving FD solution to high order PDE in the usual way can be time consuming. One may observe that the FD coefficients in the numerator possess the same magnitude as elements on the Pascal's Triangle, whereby each row n of the Pascal's Triangle correspond to the order n of the PDE, and every element r on the same row in Pascal's Triangle corresponds to the magnitude of the FD coefficients. This suggests that one may write down the FD solution to PDE using Pascal's Triangle as a guide, hence saving time and effort. Two discrepancies are observed, however:

(i) The coefficients have alternating signs as r increases from 0 to n; and

(ii) There exists a factor of $(\delta x)^{-n}$ in the FD equation corresponding to PDE of order n.

Since Pascal's Triangle can be expressed in terms of the binomial coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}, \tag{1}$$

the coefficients for forward and backward difference equations can be expressed as

$$\frac{(-1)^r}{(\delta x)^n}\binom{n}{r} \tag{2}$$

to make up for the above-mentioned discrepancies. The next step would be to assign the coefficients to the quantity ϕ at the FD grid points. We note that:

Application of Pascal's Triangle

(i) For forward difference, the FD grid points begin with $(i+n)$ for $r=0$ and decrease to i for $r=n$; and

(ii) For backward difference, the FD grid points begin with i for $r=0$ and decrease to $(i-n)$ for $r=n$.

Considering (i) and (ii), the grid points for forward and backward difference equations can be assigned as ϕ_{i+n-r} and ϕ_{i-r} respectively. Hence the forward difference and backward difference solutions to PDE can be written in terms of binomial coefficients as

$$\frac{\partial^n \phi_i}{\partial x^n} = \frac{1}{(\delta x)^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \phi_{i+n-r} \qquad \text{(forward)} \tag{3}$$

and

$$\frac{\partial^n \phi_i}{\partial x^n} = \frac{1}{(\delta x)^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \phi_{i-r} \qquad \text{(backward)} \tag{4}$$

respectively.

Table 3 shows the central difference solution of PDE derived in the same manner as forward and backward difference equations.

Table 3 Central difference equations for even-ordered partial differentials.

n	Partial differential	Central difference equations
0	$\phi_i =$	$\dfrac{\mathbf{1}\phi_i}{(\delta x)^0}$
1	$\dfrac{\partial \phi_i}{\partial x} =$	$\dfrac{\mathbf{1}\phi_{i+\frac{1}{2}} - \mathbf{1}\phi_{i-\frac{1}{2}}}{(\delta x)^1}$
2	$\dfrac{\partial^2 \phi_i}{\partial x^2} =$	$\dfrac{\mathbf{1}\phi_{i+1} - \mathbf{2}\phi_i + \mathbf{1}\phi_{i-1}}{(\delta x)^2}$
3	$\dfrac{\partial^3 \phi_i}{\partial x^3} =$	$\dfrac{\mathbf{1}\phi_{i+\frac{3}{2}} - \mathbf{3}\phi_{i+\frac{1}{2}} + \mathbf{3}\phi_{i-\frac{1}{2}} - \mathbf{1}\phi_{i-\frac{3}{2}}}{(\delta x^3)}$
4	$\dfrac{\partial^4 \phi_i}{\partial x^4} =$	$\dfrac{\mathbf{1}\phi_{i+2} - \mathbf{4}\phi_{i+1} + \mathbf{6}\phi_i - \mathbf{4}\phi_{i-1} + \mathbf{1}\phi_{i-2}}{(\delta x)^4}$
5	$\dfrac{\partial^5 \phi_i}{\partial x^5} =$	$\dfrac{\mathbf{1}\phi_{i+\frac{5}{2}} - \mathbf{5}\phi_{i+\frac{3}{2}} + \mathbf{10}\phi_{i+\frac{1}{2}} - \mathbf{10}\phi_{i-\frac{1}{2}} + \mathbf{5}\phi_{i-\frac{3}{2}} - \mathbf{1}\phi_{i-\frac{5}{2}}}{(\delta x)^5}$

However, the so-called central difference solutions to odd-ordered PDEs shown in Table 3 are invalid as the solution falls on intermediate points, i.e. $i \pm \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ etc.

Application of Pascal's Triangle

Hence coefficients of central difference solution to PDE as shown in Equation (2) are applicable only to even-ordered differentials. For these FD equations, the grid points begin with $\left(i+\frac{n}{2}\right)$ at $r=0$ and decreases to $\left(i-\frac{n}{2}\right)$ at $r=n$. So, a suitable assignment of central difference coefficients to grid points would be $\phi_{i+(\frac{n}{2})-r}$. As such, the central difference solution to even-ordered PDE can be written as

$$\frac{\partial^n \phi_i}{\partial x^n} = \frac{1}{(\delta x)^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \phi_{i+(\frac{n}{2})-r} \quad (n = \text{even}). \tag{5}$$

3. Conclusion

An observation of Pascal's Triangle pattern in FD solution to a class of partial differentials has enabled us to describe them in terms of binomial coefficients. A summary of these FD equations is listed in Table 4.

Table 4 Summary of finite difference equations to differential forms in terms of binomial coefficients (or Pascal's Triangle).

Category	FD solution to PDE
Forward difference	$\frac{\partial^n \phi_i}{\partial x^n} = \frac{1}{(\delta x)^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \phi_{i+n-r}$
Backward difference	$\frac{\partial^n \phi_i}{\partial x^n} = \frac{1}{(\delta x)^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \phi_{i-r}$
Central difference (n = even)	$\frac{\partial^n \phi_i}{\partial x^n} = \frac{1}{(\delta x)^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} \phi_{i+(\frac{n}{2})-r}$

Since all the FD solutions to this class of PDE exhibit some form of uniformity, we may generalize

$$\frac{\partial^n \phi_i}{\partial x^n} = \frac{1}{(\delta x)^n} \sum_{r=0}^{n} \left[(-1)^r \binom{n}{r} \phi_i(n,r) \right] \tag{6}$$

where

$$\phi_i(n,r) = \begin{cases} \phi_{i+n-r}; & \text{for forward differencing} \\ \phi_{i-r}; & \text{for backward differencing} \\ \phi_{i+(n/2)-r}; & \text{for central differencing } (n = \text{even}) \end{cases} \tag{7}$$

Application of Pascal's Triangle

References

[1] N.L. Biggs. The roots of combinatorics. *Historia Math.*, vol.6, no.2, pp.109-136, 1979.

[2] H. Eves. *An Introduction to the History of Mathematics.* 5th Edition. CBS College Publishing: USA, pp.159, 1983.

[3] C.C. Clawson. *Mathematical Mysteries: The Beauty and Magic of Numbers.* Plenum Publishing Corporation: New York, pp.133, 1996.

[4] B. Pascal, *Traite de Triangle Arithmetique.* Editor: Oeuvres. Brunschwig and Boutroux: Paris, vol.3, p.463, 1923.

[5] R.C. Bollinger. Extended Pascal's Triangles. *Math. Mag.*, vol.66, no.2, pp.87-94, 1993.

[6] H. Walser. The Pascal Pyramid. *Coll. Math. J.*, vol.31, no. 5, pp.383-392, 2000.

[7] T.M. Green, Recurrent sequences and Pascal's Triangle. *Math. Mag.*, vol.41, no.1, pp.13-21, 1968.

[8] C. Cadogan. Some generalizations of the Pascal's Triangle. *Math. Mag.*, vol.45, no.3, pp.158-162, 1972.

[9] Z. Usiskin. Perfect square patterns in the Pascal Triangle. *Math. Mag.*, vol.46, no.5, pp.203-208, 1973.

[10] P. Hilton and J. Pederson. Looking into Pascal's Triangle: Combinatorics, arithmetic and geometry. *Math. Mag.*, vol.60, no.5, pp.305-316, 1987.

[11] P. Hilton and J. Pederson. Relating geometry and algebra in the Pascal Triangle, Hexagon, Tetrahedron and Cuboctahedron. Part 1: Binomial coefficients, extended binomial coefficients and preparation for futher work. *Coll. Math. J.*, vol.30, no.3, pp.170-186, 1999.

[12] P. Hilton and J. Pederson. Relating geometry and algebra in the Pascal Triangle, Hexagon, Tetrahedron and Cubotahedron. Part 2: Geometry and algebra in higher dimensions: Identifying the Pascal Cuboctahedron. *Coll. Math. J.*, vol.30, no.4, pp.279-292, 1999.

[13] B. Rasof. Pascal-type triangles for the Fourier expansions of $2^{n-1}\cos^n x$ and $2^{n-1}\sin^n x$. *Math. Mag.*, vol.40, pp.235-240, 1967.

[14] Y. Nahir. The trigonometric triangle. *Int. J. Math. Edu. Sci. Technol.*, vol.32, no.4, pp.614-624, 2001.

[15] T.C. Lim. Number patterns for arithmetical aesthetics. *Math. Medley*, vol.30, no.2, pp.125-133, 2003.

Teik-Cheng Lim
Nanoscience and Nanotechnology Initiative
Faculty of Engineering
National University of Singapore
9 Engineering Drive 1, Singapore 117576