Competition corner

Please send your solutions and all other communications about this column to

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Problems

1. (IMO 2003 shortlisted problems.) Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing A and C whose centre does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the bisector of $\angle AQC$ and the line AC intersect at a point which does not depend on the choice of Γ.

2. (Canada Mathematical Olympiad, 2003) Let S be a set of n points in the plane such that any two points of S are at least 1 unit apart. Prove there is a subset T of S with at least $n/7$ points such that any two points of T are at least $\sqrt{3}$ units apart.

3. (German National Mathematical Competition, 1st round, 2003.) Determine, with proof, the set of all positive integers that cannot be represented in the form $\frac{a}{b} + \frac{a+1}{b+1}$, where a and b are positive integers.

4. (Hong Kong Mathematical Olympiad, 2003) Let p be an odd prime such that $p \equiv 1 \pmod 4$. Evaluate, with reasons,

$$\sum_{k=1}^{\frac{p-1}{2}} \left\{ \frac{k^2}{p} \right\},$$

where $\{x\} = x - \lfloor x \rfloor$, $\lfloor x \rfloor$ being the greatest integer not exceeding x.

5. (British Mathematical Olympiad, 2003) Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation of the set $\mathbb{N}$ of all positive integers.

(i) Show that there is an arithmetic progression of positive integers $a, a+d, a+2d$, where $d > 0$, such that

$$f(a) < f(a+d) < f(a+2d).$$

(ii) Must there be an arithmetic progression $a, a+d, \ldots, a+2003d$, where $d > 0$ such that

$$f(a) < f(a+d) < \cdots < f(a+2003d)?$$

6. (Vietnam 2003) Two circles Γ_1 and Γ_2 with centres O_1 and O_2, respectively, touch each other at the point M. The radius of Γ_2 is larger that of Γ_1. A is a point on Γ_2 such that the points O_1, O_2 and A are not collinear. Let AB and AC be tangents of Γ_1 with touching points B and C. The lines MB and MC meet Γ_2 again at E and F, respectively. Let D be the point of intersection of the line EF and the tangent to Γ_2 at A. Prove that the locus of D is a straight line when A moves on Γ_2 so that O_1, O_2 and A are not collinear.

7. (Hong Kong Mathematical Olympiad, 2003) Two circles intersect at points A and B. Through the point B a straight line is drawn, intersecting the first circle at K and the second circle at M. A line parallel to AM is tangent to the first circle at Q. The line AQ intersects the second circle again at R.

(a) Prove that the tangent to the second circle at R is parallel to AK.

(b) Prove that these two tangents are concurrent with KM.

8. (Belarus Mathematical Olympiad, 2003) Let

$$f(x) = (x+1)^p(x-3)^q = x^n + a_1x^{n-1} + a_2x_{n-2} + \cdots + a_{n-1}x + a_n,$$

where p and q are positive integers.

(a) Given that $a_1 = a_2$, prove that $3n$ is a perfect square.

(b) Prove that there exist infinitely many pairs (p, q) of positive integers p and q such that the equality $a_1 = a_2$ is valid for the polynomial $p(x)$.

9. (Russia Mathematical Olympiad, 2003) Find the greatest natural number N such that for any arrangement of the natural numbers $1, 2, \ldots, 400$ in the cells of a 20×20 square table there exist two numbers located in the same row or in the same column such that their difference is not less than N.

10. (Czech and Slovak Mathematical Olympiad, 2003) Find all possible values of the expression

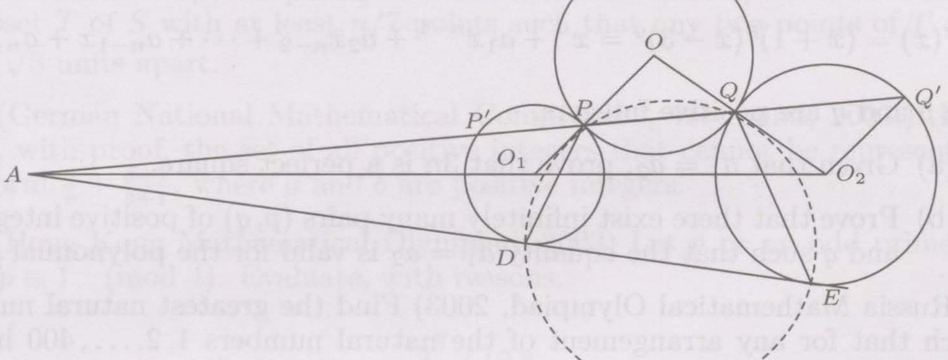
$$\frac{a^4 + b^4 + c^4}{a^2b^2 + a^2c^2 + b^2c^2},$$

where a, b, c are the lengths of the sides of a triangle.

Solutions

1. (Bulgaria Mathematical Olympiad, 1994) A triangle ABC is given with $p = \frac{1}{2}(AB + BC + CA)$. A circle k_1 touches the side BC and the sides AB, AC extended. A circle k touches k_1 and the incircle of $\triangle ABC$ at points Q and P. Let R be the point of intersection of the line PQ and the bisector of $\angle BAC$ and RT be a tangent to k. Prove that $RT = \sqrt{p(p-a)}$.

Solution by Calvin Lin Ziwei.



We first prove the following lemma.

Lemma. C_1 and C_2 are 2 non-intersecting circles. An external tangent touches C_1 at D and C_2 at E. DE and O_1O_2 meet at A. A third circle is tangent to C_1 at P, C_2 at Q. Then P, Q, A are collinear. Conversely, if a line through A intersects C_1 at P', P and C_2 at Q, Q', in that order, then there is a third circle that touches C_1 at P and C_2 at Q. Moreover, $PQED$ is cyclic.

Proof. Let the radii of C_1 and C_2 be r_1 and r_2, respectively. Let PQ intersect O_1O_2 at R and C_2 again at Q'. Then

$$\angle RPO_1 = \angle OPQ = \angle OQP = \angle O_2QQ' = \angle O_2Q'Q.$$

Therefore $\triangle RPO_1 \simeq \triangle RO_2Q'$ whence $RO_1/RO_2 = r_1/r_2$. First we note that $AO_1/AO_2 = r_1/r_2$ because $\triangle ADO_1 \simeq \triangle AEO_2$. Therefore $R = A$.

We shall now prove the converse. First we note that $AD/AE = r_1/r_2$ (because $\triangle ADO_1 \simeq \triangle AEO_2$). Thus A is the centre of the homothety that takes C_1 to C_2 with positive ratio. This homothety takes O_1 to O_2 and P to Q'. Therefore $\angle APO_1 = \angle AQ'O_2$. Let O be the intersection of O_1P with O_2Q. Then

$$\angle OPQ = \angle APO_1 = \angle AQ'O_2 = \angle O_2QQ' = \angle OQP.$$

Thus $OP = OQ$ and the circle with centre O and radius OP is the required third circle. To prove that $PQED$ is cyclic we first note that $\angle APD = \angle AQ'E$ by the homothety. By the alternate segment theorem, we get $\angle AQ'E = \angle QED$, whence $\angle APD = \angle QED$. Therefore $PQED$ is cyclic and the proof of the lemma is complete.

To solve the problem, observe that by the lemma, $A = R$. Considering the power of R with respect to the third circle, we get $RT^2 = RP \cdot RQ$. If the incircle touches the side AB at D and E, respectively, we get $AD = p - a$ and $AE = p$. Using the fact that $PQED$ is cyclic, we get $RT^2 = RP \cdot RQ = AD \cdot AE = (p-a)p$ as required.

2. (Russia 2000) A positive in n is called *perfect* if the sum of all its positive divisors excluding n itself, equals n. For example 6 is perfect because $6 = 1 + 2 + 3$. Prove that

(a) if a perfect integer larger than 6 is divisible by 3, then it is also divisible by 9.

(b) if a perfect integer larger than 28 is divisible by 7, then it is also divisible by 49.

Solution by Daniel Chen Chongli. For any positive integer n, let $\sigma(n)$ denote the sum of all its positive divisors. Thus a number n is perfect if and only if $\sigma(n) = 2n$. It's easy to prove that $\sigma(mn) = \sigma(m)\sigma(n)$ if m, n are coprime.

(a) Let $n(> 6)$ be a perfect number such that $3 \mid n$. Suppose on the contrary that $9 \nmid n$. Then, letting $n = 3a$,

$$2n = \sigma(n) = \sigma(3a) = \sigma(3)\sigma(a) = 4\sigma(a).$$

Thus $2 \mid n$. Since $n > 6$, $1, n/2, n/3, n/6$ are distinct factors of n. Thus

$$n \geq 1 + n/2 + n/3 + n/6 = n + 1,$$

a contradiction. Thus $9 \mid n$.

(b) Let $n(> 28)$ be a perfect number such that $7 \mid n$. Suppose on the contrary that $49 \nmid n$. Then, letting $n = 7a$, we have

$$2n = \sigma(n) = \sigma(7)\sigma(a) = 8\sigma(a).$$

Thus $4 \mid n$. Since $n > 28$, $1, n/28, n/14, n/7, n/4, n/2$ are distinct factors of n. Thus

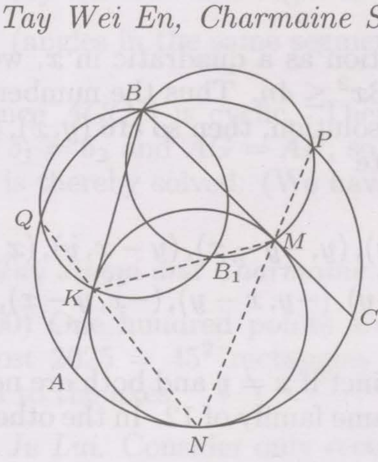
$$n \geq 1 + n/28 + n/14 + n/7 + n/4 + n/2 = n + 1,$$

a contradiction. Thus $49 \mid n$.

Also solved by A. Robert Pargeter, who pointed out that (1) no one has yet discovered an odd perfect number but it is known that if such exist they exceed 10^{300}; and (2) all even perfect numbers are of the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is prime.

3. (Russia 2000) Circles ω_1 and ω_2 are internally tangent at N, with ω_1 larger than ω_2. The chords BA and BC of ω_1 are tangent to ω_2 at K and M, respectively. Let Q and P be the midpoints of the arcs AB and BC not containing the point N. Let the circumcircles of triangles BQK and BPM intersect at B and B_1. Prove that BPB_1Q is a parallelogram.

Similar solution by Joel Tay Wei En, Charmaine Sia Jia Min and Ong Xing Cong.



Since the tangent at Q is parallel to AB, the homothety with centre N, taking ω_2 to ω_1 takes K to Q. Thus Q, K, N are collinear. Similarly, P, M, N are collinear. So $\angle BB_1K + \angle BB_1M = 180° - \angle BQK + 180° - \angle BPM = 360° - \angle BQN - \angle BPN = 180°$, and thus K, B_1, M are collinear. Next we have $\angle BQB_1 = \angle BKB_1 = \angle BMB_1 = \angle BPB_1$ and

$$\begin{aligned}\angle PBQ &= 180° - \angle KNM = \angle MKN + \angle KMN \\ &= \angle BMP + \angle QKB = \angle BB_1P + \angle QB_1B = \angle QB_1P.\end{aligned}$$

Thus BPB_1Q is a parallelogram

4. (Taiwan 2000) Let $A = \{1, 2, \ldots, n\}$, where n is a positive integer. A subset of A is *connected* if it is a nonempty set which consists of one element or of consecutive integers. Determine the greatest integer k for which A contains k distinct subsets $A_1, A_2, \ldots, A_k$, such that the intersection of any two distinct sets A_i and A_j is connected.

Similar solution by Joel Tay Wei En and Ong Xing Cong. For any $i \neq j$, if

$$\max\{A_i\} = \max\{A_j\} \quad \text{and} \quad \min\{A_i\} = \min\{A_j\}.$$

then $A_i \cap A_j$ contains $\min\{A_i\}$ and $\max\{A_i\}$ and thus must contain all the numbers between these two, whence $A_i = A_j$. Hence the pairs $(\min\{A_i\}, \max\{A_i\})$, $(\min\{A_j\}, \max\{A_j\})$, $i \neq j$, are distinct. Moreover, $\max A_j \geq \min\{A_i\}$ otherwise their intersection is empty. So there exists m such that $\min\{A_i\} \leq m \leq \max\{A_i\}$ for all i. Therefore

$$k \leq m \times (n - m + 1) \leq \lfloor (n+1)/2 \rfloor \lceil (n+1)/2 \rceil.$$

The maximum is achieved by taking all connected sets containing the element $\lfloor (n+1)/2 \rfloor$. Thus the answer is $\lfloor (n+1)/2 \rfloor \lceil (n+1)/2 \rceil$.

5. (Turkey 2000)

(a) Prove that for each positive integer n, the number of ordered pairs (x, y) of integers satisfying

$$x^2 - xy + y^2 = n$$

is finite and divisible by 6.

(b) Find all ordered pairs (x, y) of integers satisfying

$$x^2 - xy + y^2 = 727.$$

Similar solution by Andre Kueh Ju Lui, Charmaine Sia Jia Min and A. Robert Pargeter.

(a) Treating the equation as a quadratic in x, we get $y^2 - 4(y^2 - n) \geq 0$. Thus $3y^2 \leq 4n$. Similarly $3x^2 \leq 4n$. Thus the number of solutions is finite. We also note that if (x, y) is a solution, then so are $(y, x), (-x, -y), (x, x-y)$. Thus from (x, y), we can generate

$$(x, y), (y, x), (x, x-y), (y, -y-x), (y-x, y), (x-y, x),$$
$$(y-x, -x), (x-y, -y), (-y, x-y), (-x, y-x), (-y, -x), (-x, -y).$$

These 12 solutions are distinct if $x \neq y$ and both are nonzero. Moreover, anyone of these will generate the same family of 12. In the other case, we have the family of six

$$(x, x), (x, 0), (-x, 0), (0, x), (0, -x), (-x, -x).$$

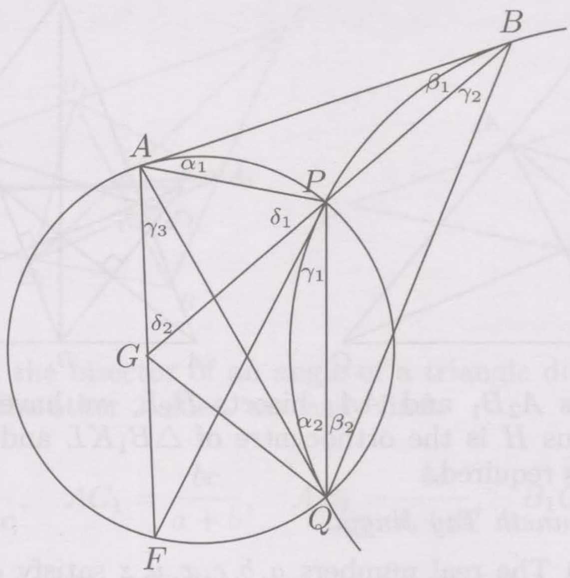
Thus the number of solutions is always a multiple of 6.

(b) From $y^2 \leq 4n/3$, we get $-32 < y < 32$. Also the discriminant $2908 - 3y^2$ must be a perfect square. This is so only for $y = 13, 18, 31$. They yield the same family:

$$(31, 13), (13, 31), (31, 18), (13, -18), (18, 31), (-18, 13),$$
$$(18, -13), (-18, -31), (-13, 18), (-31, -18), (-13, -31), (-31, -13).$$

6. (Vietnam 2000) Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at two points P and Q. The common tangent of $\mathcal{C}_1$ and $\mathcal{C}_2$ closer to P touches $\mathcal{C}_1$ at A and $\mathcal{C}_2$ at B. The tangent to $\mathcal{C}_1$ at P intersects $\mathcal{C}_2$ at E (distinct from P) and the tangent to $\mathcal{C}_2$ at P intersects $\mathcal{C}_1$ at F (distinct from P). Let H and K be two points on the rays AF and BE, respectively, such that $AH = AP$, $BK = BP$. Prove that the five points A, H, Q, K, B lie on the same circle.

Solution by A. Robert Pargeter. We are asked in effect to prove that H and K lie on the circumcircle of $\triangle AQB$. By symmetry what goes for H goes for K, so it is sufficient to prove the result for (say) H.



Forget about H and let BP produced meet AF at G. Join G, A, P and B to Q. Then, using the notation of the figure,

$$\alpha_1 = \alpha_2 \quad \text{(by the alternate segment theorem.)}$$

$$\beta_1 = \beta_2 \quad \text{(by the alternate segment theorem.)}$$

$$\gamma_1 = \gamma_2 \quad \text{(by the alternate segment theorem.)}$$

$$\gamma_1 = \gamma_3 \quad \text{(angles in the same segment)}$$

Therefore $\gamma_3 = \gamma_2$, whence $AGQB$ is cyclic. Therefore $\delta_2 = \alpha_2 + \beta_1$. Also $\delta_1 = \alpha_1 + \beta_1$. Therefore $\delta_1 = \delta_2$ and $AG = AP$, so that G is in fact the point H of the problem which is thereby solved. (We have in fact proved that BPH is a straight line.)

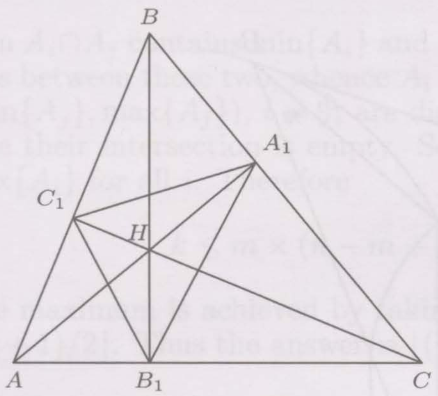
Also solved by Kenneth Tay Jingyi and Charmaine Sia Jia Min.

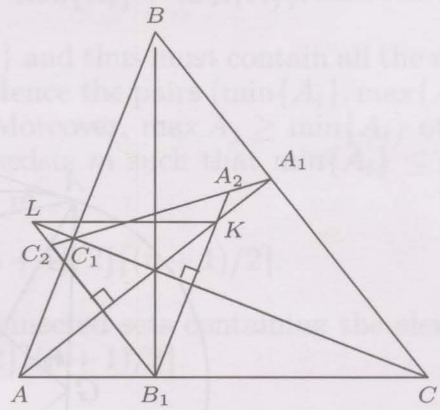
7. (St. Petersburg 2000) One hundred points are chosen in the coordinate plane. Show that at most $2025 = 45^2$ rectangles with vertices among these points have sides parallel to the axes.

Solution by Andre Kueh Ju Lui. Consider only rectangles whose sides are parallel to the axes. Suppose on the contrary that there are more than 2025 such rectangles. Let each rectangle give a value 1 to each of its vertices. Then the total value is $\geq 4 \times 2026$. Since there are 100 points, there is a point whose value is at least 82. Without loss of generality, let this point be the origin. Each of the 82 rectangles contain one point not on the axes and distinct rectangles are associated with distinct points (not on the axes). Thus there are at least 82 points not on the axes and so there are at most 17 points (other than the origin) on the axes. If there are x points on the x-axis and $17 - x$ points on the y-axis, then there are at most $x(17 - x) \leq 8 \times 9 = 72$ rectangles, a contradiction. Thus there are at most 2025 such rectangles.

8. (St. Petersburg 2000) Let AA_1, BB_1, CC_1 be the altitudes of an acute triangle ABC. The points A_2 and C_2 on line A_1C_1 are such that line CC_1 bisects the segment A_2B_1 and line AA_1 bisects the segment C_2B_1. Lines A_2B_1 and AA_1 meet at K, and lines C_2B_1 and CC_1 meet at L. Prove that lines KL and AC are parallel.

Solution by Joel Tay Wei En. We have $\angle AA_1B_1 = \angle C_1CB_1$ (since HB_1CA_1 is cyclic), $\angle AA_1C_1 = \angle B_1BC_1$ (since HA_1BC_1 is cyclic) and $\angle ABB_1 = \angle C_1CA$ (since BCB_1C_1 is cyclic). Therefore AA_1 bisects $\angle B_1A_1C_1$. Similarly, BB_1 bisects $\angle A_1B_1C_1$. Thus H is the incentre of $\triangle A_1B_1C_1$.





Since CC_1 bisects A_2B_1 and AA_1 bisects B_1L, we have $CC_1 \perp B_1A_2$ and $AA_1 \perp B_1L$. Thus H is the orthocentre of $\triangle B_1KL$ and hence $KL \perp BB_1$. Thus $KL \parallel AC$ as required.

Also solved by Kenneth Tay Jingyi.

9. (Korea 2000) The real numbers a, b, c, x, y, z satisfy $a \geq b \geq c > 0$ and $x \geq y \geq z > 0$. Prove that

$$\frac{a^2x^2}{(by+cz)(bz+cy)} + \frac{b^2y^2}{(cz+ax)(cx+az)} + \frac{c^2z^2}{(ax+by)(ay+bx)} \geq \frac{3}{4}.$$

Similar solution by Charmaine Sia Jia Min and Kenneth Tay Jingyi. Since $a \geq b \geq c > 0$ and $x \geq y \geq z > 0$, we have, by the rearrangement inequality, $by + cz \geq bz + cy$, etc. Thus

$$\text{LHS} \geq \frac{(ax)^2}{(by+cz)^2} + \frac{(by)^2}{(cz+ax)^2} + \frac{(cz)^2}{(ax+by)^2}.$$

Letting $p = ax, q = by, r = cz$, we have $p \geq q \geq r > 0$ and $p^2 + q^2 \geq p^2 + r^2 \geq q^2 + r^2$. By the power mean inequality and then the rearrangement inequality, we get

$$\frac{p^2}{(q+r)^2} + \frac{q^2}{(p+r)^2} + \frac{r^2}{(p+q)^2} \geq \frac{p^2}{2(q^2+r^2)} + \frac{q^2}{2(p^2+r^2)} + \frac{r^2}{2(p^2+q^2)}$$

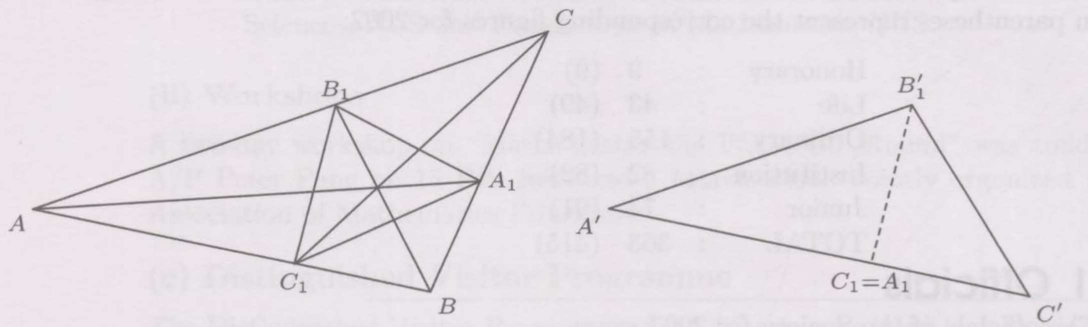
$$= \frac{1}{4}\left(\frac{2p^2}{q^2+r^2} + \frac{2q^2}{p^2+r^2} + \frac{2r^2}{p^2+q^2}\right) \geq \frac{1}{4}\left(\left[\frac{q^2}{q^2+r^2} + \frac{r^2}{p^2+r^2} + \frac{p^2}{p^2+q^2}\right]\right.$$

$$\left. + \left[\frac{r^2}{q^2+r^2} + \frac{p^2}{p^2+r^2} + \frac{q^2}{p^2+q^2}\right]\right) = \frac{3}{4}.$$

10. (Mongolia 2000) The bisectors of angles A, B, C of $\triangle ABC$ intersect its sides at points A_1, B_1, C_1. Prove that if the quadrilateral $BA_1B_1C_1$ is cyclic, then

$$\frac{BC}{AC+AB} = \frac{AC}{AB+BC} - \frac{AB}{BC+AC}.$$

Solution by A. Robert Pargeter. If $BA_1B_1C_1$ is cyclic, then $\angle B_1C_1A_1 = \angle B_1BA_1 = \angle C_1BB_1 = \angle C_1A_1B_1$. Therefore $B_1C_1 = B_1A_1$; and $\angle AC_1B = \angle BA_1B_1$. Thus the triangles AB_1C_1, CB_1A can be fitted together to make a single triangle (see figure).



Using theorem that the bisector of an angle of a triangle divides the opposite side in the ratio of the other 2 sides, we easily find:

$$A_1C = \frac{ab}{b+c}, \quad AC_1 = \frac{bc}{a+b}, \quad AB_1 = \frac{bc}{c+a}, \quad B_1C = \frac{ab}{c+a}$$

Clearly the "new" triangle $A'B_1'C'$ is equiangular to ABC. Therefore

$$\frac{A'C'}{A'B_1'} = \frac{AC}{AB} = \frac{b}{c}$$

$$\text{i.e.,} \quad \frac{bc}{a+b} + \frac{ab}{b+c} = \frac{b}{c} \times \frac{bc}{c+a}$$

$$\text{i.e.,} \quad \frac{c}{a+b} + \frac{a}{b+c} = \frac{b}{c+a}$$

which is what we want to prove.

Also solved by Kenneth Tay Jingyi.