

USING STEWART'S THEOREM

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Introduction. In second author's ninth grade geometry text [1], Stewart's Theorem is relegated to pages 721 through 723 in the last chapter which is headed "Enrichment Topics." Since there are so many topics that have to be covered in a ninth grade geometry class that begins the secondary curriculum leading to Advanced Placement Calculus, he has always run out of time well before reaching page 721. Consequently, he has never developed or presented the theorem in his geometry class. It is a lovely theorem about which he and, perhaps, other geometry teachers, need to remind themselves from time to time. It is easy to prove (as opposed to discovering on one's own), and Posamentier's *Excursions in Advanced Euclidean Geometry* provides a proof and nice uses of the theorem [2].

The theorem was known to mathematicians of the Fourth Century A.D., but the theorem's eponym was the Eighteenth Century Scottish mathematician Matthew Stewart [1]. Investigations and applications of Stewart's Theorem would provide interesting and suitable projects for ninth grade students who wish to do a little extra geometry. In the work to follow, we shall state Stewart's Theorem without proof, give a simple example of its use, and then apply it to an interesting problem which has come to our attention.

Stewart's Theorem. Let us consider $\triangle ABC$ with cevian \overline{CD} as shown in Figure 1 and denote the lengths of relevant segments in the following manner:

$$AB = c, \quad BC = a, \quad AC = b, \quad CD = d, \quad BD = m, \quad \text{and} \quad AD = n.$$

The claim made by Stewart's Theorem is that

$$a^2n + b^2m = c d^2 + cmn.$$

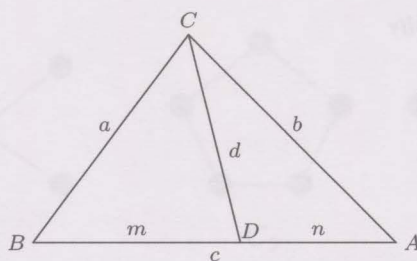


Figure 1 $\triangle ABC$ with cevian \overline{CD} .

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The proof is straightforward and can be accomplished with two applications of the Pythagorean Theorem [1,2].

Example. Suppose that in $\triangle ABC$, $AB = c = 8$, $BC = a = 6$, $AC = b = 7$, $BD = m = 3$, and $AD = n = 5$. Let us find the value of $CD = d$.

First Solution. We make use of Stewart's Theorem by writing $6^2(5) + 7^2(3) = 8 d^2 + 8(3)(5)$ which implies that $d^2 = 207/8$. Thus $d = \sqrt{207/8} = (3/2)\sqrt{23/2} \approx 5.08675$. \square

Second Solution. Perhaps, Stewart's Theorem, elegant as it is, has been relegated to the "enrichment" chapter of the geometry text because examples such as this one can be solved with trigonometry. In this case, two applications of the law of cosines will do quite nicely. We see that in $\triangle ABC$.

$$\cos B = (a^2 + c^2 - b^2)/(2ac) = (36 + 64 - 49)/96 = 17/32$$

and that in $\triangle DBC$

$$d^2 = a^2 + m^2 - 2am \cos B = 36 + 9 - 36(17/32) = 207/8.$$

Thus $d = \sqrt{207/8}$ as previously obtained. \square

The interesting problem to which we referred is presented below. It was stated and solved by Professor Željko Hanjš of the University of Zagreb, Croatia, who shared his work with the first author.

Problem. It is given that $CB > CA$ in $\triangle ABC$ and that cevian \overline{CT} bisects $\angle ACB$. Furthermore, cevian \overline{CS} is given with $BS = TA$ as shown in Figure 2.

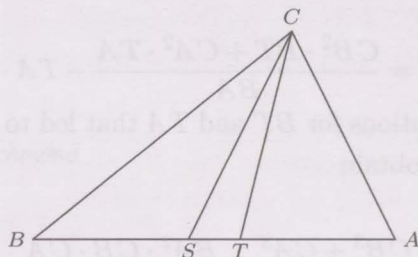


Figure 2. $\triangle ABC$ with cevians \overline{CT} and \overline{CS} .

Prove that $CS^2 - CT^2 = (CB - CA)^2$.

Proof. Since T is between B and A , we may write that $BT + TA = BA$. Since \overline{CT} bisects $\angle ACB$, we may also write that $BT/TA = CB/CA$. It follows that

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$$BT/(BA - BT) = CB/CA. \text{ Therefore, } BT = \frac{BA \cdot CB}{CA + CB} \text{ and } TA = BA - BT = \frac{BA \cdot CA}{CA + CB}.$$

We are now in a position to apply Stewart's Theorem to cevian \overline{CT} in $\triangle ABC$. We may write that

$$CB^2 \cdot TA + CA^2 \cdot BT = BA \cdot CT^2 + BA \cdot BT \cdot TA$$

which implies that

$$CT^2 = \frac{CB^2 \cdot TA + CA^2 \cdot BT}{BA} - BT \cdot TA.$$

Next, we substitute for BT and TA the expressions in terms of the sides of $\triangle ABC$ as found above. The substitution followed by a bit of algebra yields

$$(1) \quad CT^2 = CB \cdot CA - \frac{BA^2 \cdot CB \cdot CA}{(CA + CB)^2}.$$

Let us now turn our attention to cevian \overline{CS} and apply Stewart's Theorem again, this time to \overline{CS} in $\triangle ABC$. We may write that

$$CB^2 \cdot SA + CA^2 \cdot BS = BA \cdot CS^2 + BA \cdot BS \cdot SA.$$

This equation implies that

$$(2) \quad CS^2 = \frac{CB^2 \cdot SA + CA^2 \cdot BS}{BA} - BS \cdot SA.$$

We have been given that $BS = TA$ for this problem. Therefore, $SA = BT$ as well, and equation (2) becomes

$$CS^2 = \frac{CB^2 \cdot BT + CA^2 \cdot TA}{BA} - TA \cdot BT.$$

Making the same substitutions for BT and TA that led to equation 1 and doing the same sort of algebra, we obtain

$$(3) \quad \begin{aligned} CS^2 &= \frac{CB^3 + CA^3}{CA + CB} - \frac{BA^2 \cdot CB \cdot CA}{(CA + CB)^2} \quad \text{or} \\ CS^2 &= CB^2 - CB \cdot CA + CA^2 - \frac{BA^2 \cdot CB \cdot CA}{(CA + CB)^2}. \end{aligned}$$

Subtracting equation (1) from equation (3), we obtain

$$CS^2 - CT^2 = CB^2 - 2CB \cdot CA + CA^2 = (CB - CA)^2$$

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which is the desired result. \square

Conclusions and Another Problem. As suggested above, Stewart's Theorem is not "needed" for computation by students who know the law of cosines. However, such a nice result as the conclusion of Professor Hanjš's problem would not seem to be obtainable in so straightforward a manner by trigonometric means. It is a pity that such a lovely theorem is not more widely taught in plane geometry classes in secondary schools in America and Singapore.

We leave for our readers a problem taken from our second reference. We hope that they will agree that Stewart's Theorem provides an elegant way to attack the problem that follows:

Problem. Prove that, in a right triangle, the sum of the squares of the distances from the vertex of the right angle to the trisection points of the hypotenuse is equal to $5/9$ of the square of the length of the hypotenuse.

References

- [1] Richard Rhoad, George Milauskas, and Robert Whipple, *Geometry for Enjoyment and Challenge*, McDougal, Littell & Company, Evanston, IL (191), pp. 721-723.
- [2] Alfred S. Posamentier, *Excursions in Advanced Euclidean Geometry* (Revised Edition), Addison-Wesley Publishing Company, Reading, MA (1984), pp. 77-79.

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