Mathematical Medley Problems Corner

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A. Prized Problems

Problem 1 (Book voucher up to $150)

Proposed by Albert F.S. Wong, Temasek Polytechnic

For what values of p and q would the zeros of $x^3 + px^2 + qx + 3p = 0$ be positive integers?

Problem 2 (Book voucher up to $150)

Let $\{a_n\}$ be a sequence of positive integers satisfying the following property

$$\sum_{d|n} a_d = 2^n.$$

Prove that n divides a_n.

B. Instruction

(1) Prizes in the form of book vouchers will be awarded to one or more received best solutions submitted by secondary school or junior college students in Singapore for each of these problems.

(2) To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

(3) Solutions should be sent to : The Editor, Mathematics Medley, c/o Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore117543 ; and should arrive before 30 November 2005.

(4) The Editor's decision will be final and no correspondence will be entertained.

C. Solutions to the problems of Volume 31 No.2 2004

Problem 1

Find the exact value of the sum

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(4n+1)}.$$

Solution to Problem 1 *By Zhao Yan - Raffles Junior College*

Firstly,

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(4n+1)} = \sum_{n=1}^{\infty} \left(\frac{2}{4n+1} - \frac{1}{2n+1} \right) = 2 \sum_{n=1}^{\infty} \left(\frac{1}{4n+1} - \frac{1}{4n+2} \right)$$

On the other hand,

$$\frac{1-x}{1-x^4} = (1-x)(1+x^4+x^8+\cdots) = 1 - x + x^4 - x^5 + x^8 - x^9 + \cdots.$$

So,

$$\int_0^1 \frac{1-x}{1-x^4} dx = \int_0^1 \sum_{n=0}^{\infty} (x^{4n} - x^{4n+1}) dx = \sum_{n=0}^{\infty} \int_0^1 (x^{4n} - x^{4n+1}) dx. \qquad (*).$$

The left hand side of $(*)$ gives:

$$\begin{aligned}
\int_0^1 \frac{1-x}{(1-x)(1+x)(1+x^2)} dx &= \int_0^1 \frac{1}{(1+x)(1+x^2)} dx \\
&= \frac{1}{2} \int_0^1 \left(\frac{1}{1+x} + \frac{1-x}{1+x^2} \right) dx \\
&= \frac{1}{2} \int_0^1 \left(\frac{1}{1+x} + \frac{1}{1+x^2} - \frac{x}{1+x^2} \right) dx \\
&= \frac{1}{2} \left[\ln(1+x) + \tan^{-1} x - \frac{1}{2} \ln(1+x^2) \right]_0^1 \\
&= \frac{1}{2} \left(\frac{1}{2} \ln 2 + \frac{\pi}{4} \right).
\end{aligned}$$

The right hand side of (*) gives

$$\left[\sum_{n=0}^{\infty} \left(\frac{x^{4n+1}}{4n+1} - \frac{x^{4n+2}}{4n+2} \right) \right]_0^1 = \sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{1}{4n+2} \right).$$

Hence,

$$\sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{1}{4n+2} \right) = \frac{1}{2} \left(\frac{1}{2} \ln 2 + \frac{\pi}{4} \right) \Leftrightarrow 2 \sum_{n=0}^{\infty} \left(\frac{1}{4n+1} - \frac{1}{4n+2} \right) = \frac{1}{2} \ln 2 + \frac{\pi}{4}.$$

So,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(2n+1)(4n+1)} &= 2\sum_{n=1}^{\infty}\left(\frac{1}{4n+1}-\frac{1}{4n+2}\right)\\ &= \tfrac{1}{2}\ln 2+\tfrac{\pi}{4}-2(1-\tfrac{1}{2})=\tfrac{1}{2}\ln 2+\tfrac{\pi}{4}-1.\end{aligned}$$

Editor's note:
Zhao Yan won a book prize of $100. Note that in the solution above, interchanging of summation and integration took place in equation (*). This ia allowed provided that the integrand is uniformly bounded, which is the case. The following is an alternative solution by the contributor of the problem, Mr Albert Wong.

Observe that if $\alpha > 0$, then

$$\int_0^\infty e^{-\alpha x}\,dx=\frac{1}{\alpha}.$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(2n+1)(4n+1)} &= \tfrac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n+1/4}-\frac{1}{n+1/2}\right)\\ &= \tfrac{1}{2}\sum_{n=1}^{\infty}\int_0^\infty\left(e^{-(n+1/4)x}-e^{-(n+1/2)x}\right)dx.\end{aligned}$$

Since the integrand is uniformly bounded, we can interchange the two operations. Hence arriving at

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(2n+1)(4n+1)} &= \tfrac{1}{2}\int_0^\infty\left(\sum_{n=1}^{\infty}e^{-nx}\right)\left(e^{-x/4}-e^{-x/2}\right)dx.\\ &= \tfrac{1}{2}\int_0^\infty\tfrac{e^{-x}}{1-e^{-x}}\left(e^{-x/4}-e^{-x/2}\right)dx\end{aligned}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n+1)(4n+1)}=\frac{1}{2}\int_0^\infty\frac{e^{-5x/4}-e^{-3x/2}}{1-e^{-x}}dx. \qquad (1).$$

Applying the transformation, $e^{-x}=t^4$ to (1) yields the result

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(2n+1)(4n+1)} &= \tfrac{1}{2}\int_0^\infty\tfrac{t^5-t^6}{1-t^4}\left(\tfrac{4}{t}dt\right)\\ &= \int_0^1\left(\tfrac{1}{t+1}-\tfrac{t}{t^2+1}+\tfrac{1}{t^2+1}+2t-2\right)dt\\ &= \ln\sqrt{2}+\tfrac{\pi}{4}-1.\end{aligned}$$

Problem 2.

Let $n \geq 4$ and A be a subset of the open interval $(0, 2n)$ with n distinct elements.

Can we always find elements in A whose sum is divisible by $2n$?

Solution to Problem 2

By Kanav Arora - Anderson Junior College

The set of n integers has to be chosen from the set

$$\{1, 2, 3(n-1), n, (n+1)(2n-1)\}.$$

Now, we can see that actually all the numbers except n form a pair whose sum is equal to $2n$.

$$\begin{aligned} 1 + (2n-1) &= 2n \\ 2 + (2n-2) &= 2n \\ 3 + (2n-3) &= 2n \\ . \qquad & \quad . \\ . \qquad & \quad . \\ . \qquad & \quad . \\ (n-1) + (n+1) &= 2n \end{aligned}$$

These are a total of $(n-1)$ pairs. The presence of even one of these pairs in Set A will ensure that we can always find elements in it whose sum is divisible by $2n$.

Now, the way we choose n integers can be divided into two cases: when n is chosen and when n is not chosen.

Case 1: When n is not chosen.

Now, since there are only $(n-1)$ pairs and we have to choose n numbers from them, therefore now the Set A will have at least one complete pair whose sum is equal to $2n$. Therefore, in this case, we can always find integers whose sum is equal to $2n$.

Case 2: When n is chosen in Set A

Now, since one number chosen is n, therefore total of $(n-1)$ more numbers are left to be chosen. There are still $(n-1)$ pairs of numbers left whose sum is equal to $2n$. So, at most, the remaining $(n-1)$ numbers will be chosen such that all the numbers belong to different pairs and in doing so, all the pairs will be used up. If this is not strictly followed then we will end up with one pair in Set A and hence the condition will be satisfied. Now, we try to choose numbers such that the condition is not satisfied and then finally try to show that this actually is not possible.

First, we write all the pairs whose sum adds up to $2n$.

$$
\begin{gathered}
1, 2n-1 \\
2, 2n-2 \\
3, 2n-3 \\
4, 2n-4 \\
\cdot \\
\cdot \\
\cdot \\
n-2, n+2 \\
n-1, n+1
\end{gathered}
$$

Now, as earlier stated, we need to select only one number from each pair. Let this set of $(n-1)$ integers be defined as Set B. Since n is already present in Set A, therefore, the elements of B should be such that they should neither add up to an even multiple of n (directly satisfying the condition) nor odd multiple of n (adding n already present in Set A will make it a multiple of $2n$).

This case can be further subdivided into two cases: when 1 is chosen and when $(2n-1)$ is chosen.

Case 2(a): When 1 is chosen.

If 1 is chosen then from the last pair $(n-1)$ cannot be chosen (or else they would add up to n, i.e. is a multiple of n). Hence, the number chosen from the last pair would be $(n+1)$. Now, these numbers add up to give $(n+2)$. Now, from the second pair, $(2n-2)$ cannot be chosen (or else $(n+2)+(2n-2)=3n$). Hence, the number chosen from that pair would be 2. Now, adding 1 and 2, we get 3. Now, from the third pair $(2n-3)$ cannot be chosen (or else they would add up to give a multiple of n). Hence, the number chosen from that pair should be 3.

Similarly, adding 1 and 3 would give us 4. Hence, from the fourth pair, the number selected should be 4.

Similarly, from every pair the smaller number will be selected except for the last pair from which $(n+1)$ has already been selected. This means from the second last pair, $(n-2)$ has been selected. But if we add this to 2 which has been selected from the second pair, we get n, which is actually a multiple of n. Hence; we see that the condition cannot be satisfied. But for this, we need to ensure that the numbers 2 and $(n-2)$ are actually not the same. So, we get the inequality. Therefore, the condition is true for this case for $n > 4$. But, coincidentally, at $n = 4, 1+2+(n+1) = 1+2+5 = 8$ which actually sums up to a multiple of $2n$ or 8. Hence, the initial proposition is true in this case for $n \geq 4$.

Case 2(b): When $(2n-1)$ is chosen from the first pair.

The proof for this case is quite similar to the previous case. If $(2n-1)$ is chosen from the first pair, then from the last pair $(n-1)$ is the only valid option. Adding both of them, we get $(2n-2)$. Hence, from the second pair, $(2n-2)$ is chosen. Like this, we continue as in the previous case and we find that from every pair except the last one, the larger of the two numbers is chosen. Then, if we add up $(2n-2)$ from the second pair and $(n+2)$ from the second last pair, we get $3n$ that is a multiple of n. Hence the condition cannot be satisfied. But in this case too, we need to ensure that the above-mentioned numbers are actually different. Therefore, we get the inequality. But, in this case too, coincidentally for $n = 4$, $(2n-1)+(n-1)+(2n-2) = 7+3+6 = 16$ (a multiple of n i.e. 4). Therefore, in this case too, the initial proposition is true for $n \geq 4$. Hence, the proposition is true for case 2: When n is chosen in Set A. Therefore, after combining both the cases, we see that we can always find elements from Set A such that their sum is divisible by $2n$.

Editor's note:
Also solved correctly by Bryan Hooi Kuen-Yew from Anglo-Chinese School (Independent). The book prize of $150 is shared between Kanav and Bryan.