Mathematical Medley Problems Corner

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A. Prized Problems

1. Find all the real solution pairs (x, y) that satisfy the system

$$\begin{aligned} \frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{y}} &= (x+3y)(3x+y) \\ \frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{y}} &= 2(y^2 - x^2). \end{aligned}$$

(Note: $\sqrt{x}$ denotes the *positive* square root of the real (nonnegative) number x).

Proposed by Albert Wong, Temasek Polytechnic.

2. (a) Prove that for any positive integer n,

$$\sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n}{4k} = 2^{n-2} + (\sqrt{2})^{n-2} \cos \frac{n\pi}{4}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

(b) Prove that

$$\sum_{k=1}^{45} \tan^2(2k-1)^\circ = 4005.$$

(Note: Solution by direct algorithmic computation will not be accepted)

B. Instruction

(1) Prizes in the form of book vouchers will be awarded to one or more received best solutions submitted by secondary school or junior college students in Singapore for each of these problems.

(2) To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

(3) Solutions should be sent to : The Editor, Mathematics Medley, c/o Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore117543 ; and should arrive before 15 June 2006.

(4) The Editor's decision will be final and no correspondence will be entertained.

Solutions to the problems of Volume 32 No.1 2005

Problem 1

For what values of p and q would the zeros of $x^3 + px^2 + qx + 3p = 0$ be positive integers?

Solution to Problem 1. *By Lock Hon Mun - Raffles Junior College*

Let the roots of the equation $x^3 + px^2 + qx + 3p = 0$ be a, b and c, which are three positive integers. Then it is clear that

$$a + b + c = -p \tag{1}$$

$$abc = -3p \tag{2}$$

and

$$ab + ac + bc = q \tag{3}$$

From (1) and (2), we have

$$a + b + c = \frac{abc}{3} \tag{4}$$

Without loss of generality, we may assume that $a \geq b \geq c$. Notice that c can only take the values of 1, 2 or 3; since if $c > 3$, then $\frac{abc}{3} > 3a \geq a + b + c$, which contradicts (4).

<u>Remark:</u> It is not difficult to show that given a cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

the sum of roots is $-\frac{b}{a}$, the sum of product of roots taken in pairs is $\frac{c}{a}$ and the product of roots is $-\frac{d}{a}$.

Case 1: $c = 3$.

We claim that $b = 3$. Otherwise if $b > 3$, then by the same argument as above we have $\frac{abc}{3} > 3a \geq a + b + c$, which contradicts (4) above. By substituting the values of $b = 3$ and $c = 3$ in equation (4), we obtain $a = 3$. So $a = 3, b = 3$ and $c = 3$ is the only possible solution to the cubic equation $x^3 + px^2 + qx + 3p = 0$. Substitute the values of a, b and c into equation (1) and (3) we obtain $p = -9$ and $q = 27$.

Case 1: $c = 2$.

We claim that that b can only take the values $2, 3$ or 4. Observe that if $b \geq 5$, then $\frac{abc}{3} \geq \frac{10}{3}\ a > 3a \geq a + b + c$, which contradicts equation (4).

Substituting $b = 2$ and $c = 2$ into equation (3) we obtain $a = 12$.
Substituting $b = 3$ and $c = 2$ into equation (3) we obtain $a = 5$.
Substituting $b = 4$ and $c = 2$ into equation (3) we obtain $a = 3.6$. As a must be an integer, this case is inadmissible.
Therefore, $(a, b, c) = (12, 2, 2)$ or $(5, 3, 2)$. Substituting these sets of solutions into equation (1) and (3), we obtain $p = -16, q = 52$ or $p = -10, q = 31$ respectively.

Case 1: $c = 1$.

We claim that b is at most 9. If $b > 9$, then $\frac{abc}{3} > 3a \geq a + b + c$, which contradicts equation (4). For each possible value of b from 1 to 9, we solve a by using equation (4). But as a must be a positive integer, the only possible values for b are $4, 5$ and 6. Thus, $(a, b, c) = (15, 4, 1)$, $(9, 5, 1)$ or $(7, 6, 1)$ and the respective values of (p, q) are $(-20, 79)$, $(-15, 59)$ and $(-14, 55)$.

Editor's note:
Similar and complete solutions to the above were received from Kanav Anora (Anderson Junior College) and Zhao Yan (Raffles Junior College) . They will each receive a book prize of $50.

Problem 2

Let $\{a_i\}$ be a sequence of positive integers satisfying the following property

$$\sum_{d|n} a_d = 2^n.$$

Prove that n divides a_n.

Solution to Problem 2. *By Bryan Hooi Kuen - Anglo-Chinese School Independent*
It is clear that by the property assumed, $a_1 = 2$ and $a_2 = 2$. Obviously, $n|a_n$ when $n = 1, 2$. Therefore it suffices to consider the case when $n > 2$.

For any positive integer $n > 1$, $n = \Pi_{i=1}^{k}\ p_i^{\alpha_i}$ for some distinct primes p_i's. We then define the *length* of n, denoted by $\ell(n)$, to be the positive integer $\sum_{i=1}^{k}\ \alpha_i$.

We show by induction on $\ell(n)$ that n divides a_n. Suppose $\ell(n) = 1$. Then n is a prime. So, by Fermat's Little Theorem, $a_n = 2^n - a_1 = 2^n - 2 \equiv 0 \pmod{n}$. Thus $n|a_n$.

Next we assume that n divides a_n when $\ell(n) < m$ for some positive integer m. Suppose $\ell(n) = m$ and $n = \Pi_{i=1}^{k}\ p_i^{\alpha_i}$. Observe that for any p_j, we can express a_n as follows:

$$\begin{aligned} a_n &= 2^n - \sum_{d|n, d\neq n} a_d \\ &= 2^n - a_{\frac{n}{p_j}} - \sum_{d|n, d\neq n, d\neq \frac{n}{p_j}} a_d \\ &= 2^n - 2^{\frac{n}{p_j}} - \sum_{d|n, d\neq n, d\neq \frac{n}{p_1}} a_d + \sum_{d|\frac{n}{p_j}, d\neq \frac{n}{p_j}} a_d \\ &= \underbrace{2^{\frac{n}{p_j}}(2^{n-n/p_i} - 1)}_{A} - \underbrace{\sum_{d\,|\,n,\, p^{\alpha_j}\,|\,d,\, d\neq n} a_d}_{B}. \end{aligned}$$

Note that in A above, $n - \dfrac{n}{p_j} = p_j^{\alpha_j - 1}(p_j - 1)\Pi_{i=2}^{k} p_i^{\alpha_i} = \phi(p_j^{\alpha_j})\Pi_{i=2}^{k} p_i^{\alpha_i}$.

If $p_j = 2$, then since $\alpha_j \leq 2^{\alpha_j - 1}$ when $\alpha_1 \geq 1$, we have $p_j^{\alpha_j}$ divides A. If $p_j > 2$, then $2^{n - \frac{n}{p_1}} = (2^{\phi(p_j^{\alpha_j})})^{\Pi_{i=2}^{k} p_i^{\alpha_i}} \equiv 1 (\text{mod } p_j^{\alpha_j})$. So A is still divisible by $p_1^{\alpha_1}$.

For the term B in the above expression, any divisor d of n with $d < n$ has length less than m. By induction hypothesis, we conclude d divides a_d. Since for each of the term a_d in the summand B, $p_j^{\alpha_j}$ divides d. Hence, each term in B is also divisible by $p_j^{\alpha_j}$. In conclusion, we have $p_j^{\alpha_j}$ divides a_n.

Since j can be any integer from 1 to k and $p_j^{\alpha_j}$'s are all relatively prime, we conclude that the product $n = \Pi_{i=1}^{k} p_i^{\alpha_i}$ divides a_n.

Remark:
Compared to Problem 1, we receive relatively fewer correct solutions for this question. Similar solution was provided by Wang Tengyao (Hwa Chong Institution). They will each receive $50 book prize.