Kings in Tournaments

by
Yu Yibo, Di Junwei, Lin Min

Abstract. Landau, a mathematical biologist, showed in 1953 that any tournament T always contains a king. A king, however, may not exist in the resulting digraph $T - \{e\}$ obtained by deleting an arc e from T. In this note, we characterize those arcs e in T such that $T - \{e\}$ contains a king.

1. Tournaments

A *tournament* is a non-empty finite set of vertices in which every two vertices are joined by one and only one arrow (such an arrow is also called an *arc* or a *directed edge*). A tournament with five vertices is shown in Figure 1.

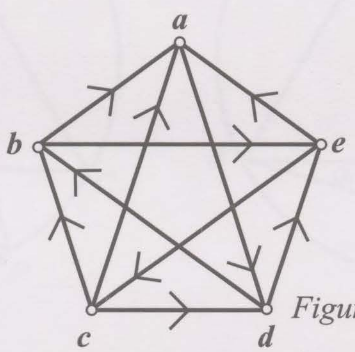

Figure 1

Such a mathematical model is called a tournament since it can be used to show the possible outcomes of a round-robin tournament. In a round-robin tournament, there is a set of players (or teams) where any two players (or teams) engage in a game that cannot end in a tie, and every player (or team) must play each other once and exactly once.

2. Terminology

Let T be a tournament and x, y be two vertices in T. If there is an arrow from x to y, we say that x **dominates** y or y is **dominated** by x (symbolically, $x \to y$). The number of vertices dominated by x is the **out-degree** of x, and is denoted by $d^+(x)$. The number of vertices that dominate x is the **in-degree** of x, denoted by $d^-(x)$. The out-degree of the vertex x is also known as the **score** of x. The set of vertices dominated by x is the **out-set** of x, $O(x)$; and the remaining set of vertices that dominate x is the **in-set** of x, $I(x)$.

Let x be a vertex and A be any set of vertices not containing x in T. We write $A \to x$ to indicate that every vertex in A dominates x; and $x \to A$ to mean that x dominates all the vertices in A.

Let B and C be any two disjoint sets of vertices in T. We write $B \dashrightarrow C$ to indicate that every vertex in C is dominated by some vertex in B.

For any 2 vertices x, y in T, the **distance from x to y**, denoted by $d(x, y)$, is the *minimum* number of arrows one has to follow in order to travel from x to y. Clearly, $d(x, x) = 0$ for any vertex x in T; $d(x, y) = 1$ if x dominates y; $d(x, y) \geq 2$ if x does not dominate y. Note that $d(x, y)$ may not be equal to $d(y, x)$, and we define $d(x, y) = \infty$ if y is not reachable from x.

3. Kings in Tournaments

Let T be a tournament with $n \geq 2$ vertices. A vertex x in T is called the **emperor** if $d(x, y) = 1$ for any other vertex y in T; that is, x dominates all other vertices in T. Clearly, x is the emperor if and only if $d^+(x) = n - 1$.

A vertex x in T is called a **king** if $d(x, y) \leq 2$ for any other vertex y in T; that is, for any other vertex y in T, either $x \to y$ or $x \to z \to y$ for some z in T. By definition, the emperor is a king, but not conversely.

Studying dominance relations in certain animal societies, the mathematical biologist Landau proved in [3] the following result:

Theorem 1. In a tournament T, any vertex with the maximum score (out-degree) is always a king. □

Moon, a Canadian mathematician, proved in [4] the following:

Theorem 2. In a tournament T, any *non-emperor* vertex ν (i.e. ν is dominated by some other vertex in T) is always dominated by a king. □

As a direct consequence of Theorem 2, we have:

Corollary 3. No tournament contains exactly two kings. □

Thus, any tournament either contains exactly one king (the emperor) or at least three kings.

4. The Main Result

Let T be a tournament and e an arc in T. Suppose e is deleted from T. Does the resulting structure now contain a king (that is, a vertex from which one can reach others within 2 steps in the resulting structure)? We shall settle this problem in what follows.

Let A be a set of vertices in T. Note that the resulting structure obtained from T by deleting the vertices in A together with those arcs incident with them remains as a tournament. We denote it by $T - A$. However, the resulting structure obtained from T by deleting some arcs only is not a tournament; for convenience, we call it a **digraph**. A **king** of a digraph is similarly defined as it is defined for a tournament. Also, an arc from vertex a to vertex b is denoted by (a, b).

The objective of this note is to establish the following result.

Theorem 4. Let T be a tournament with at least 3 vertices and $e = (a, b)$ an arc in T. Let $D(= T - \{e\})$ be the digraph obtained by deleting e from T. Then D contains at least one king if and only if $d^-(a) + d^-(b) \geq 1$ in D.

Proof: [Necessity]

Suppose on the contrary that in D, $d^-(a)+d^-(b) < 1$, i.e. $d^-(a)+d^-(b) = 0$. Then $d^-(a) = 0$ and $d^-(b) = 0$. In this case, no vertex can reach a and b in D, and thus D contains no kings.

[Sufficiency]

(1) $d^+(a) = 0$ and $d^+(b) = 0$ in D.

Let z be any king of $T - \{a, b\}$ (it exists by Theorem 2). Clearly, $z \to a$ and $z \to b$, which imply that z is a king of D.

(2) $d^+(a) = 0$ and $d^+(b) > 0$ (or vice versa) in D.

Let z be a king of $T - \{a\}$.

(a) If $z = b$, then $d(b, a) = 2$, and so $z(= b)$ is a king of D.

(b) If $z \neq b$, then $z \to a$, and hence z is also a king of D.

(3) $d^+(a) > 0$ and $d^+(b) > 0$ in D.

(a) Either $d^-(a)$ or $d^-(b)$ is zero (but not both since $d^-(a)+d^-(b) \geq 1$), say $d^-(a) = 0$. Since $a \to I(b) \to b$, $d(a, b) = 2$; and hence a is a king of D.

(b) $d^-(a) > 0$ and $d^-(b) > 0$ in D.

- If there is a king z of $T - \{a, b\}$ which dominates both a and b, then z is a king of D.
- Suppose there is a king z of $T - \{a, b\}$ which dominates either a or b, but not both, say, $z \to a$ and $b \to z$. Assume z is not a king of D. Then $d(z, b) > 2$, and so $I(b) \to z$ and $O(z) \subseteq O(b)$, excluding a.

  Let y be a king of $I(b)$. Then y is a king of D since $y \to b \to O(b)$ and $y \to z \to a$ (see Figure 2).

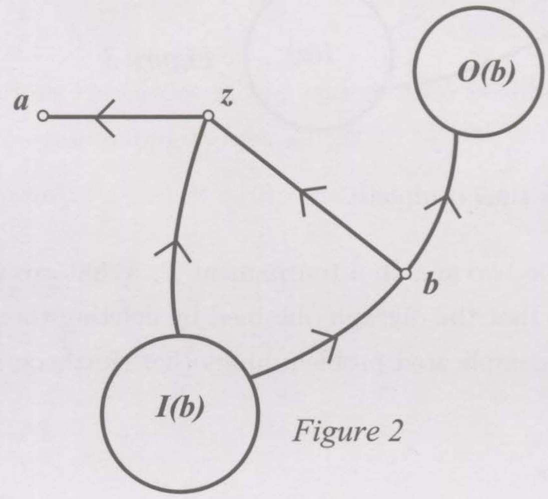

Figure 2

- Suppose all the kings z of $T - \{a, b\}$ are dominated by both a and b. Assume that there is no king of D. Then the set of vertices $S = I(a) \cap I(b)$ must dominate all such z which are not in S; otherwise, z can reach a and b through S within 2 steps, and z would be a king of D.

  If set $S \neq \emptyset$, then any king of S is a king of D since all vertices not in S are dominated by a or b.

  Thus, $S = \emptyset$ (i.e. no vertex dominates both a and b), and z cannot reach a or b or both (say, a) in two steps. Therefore $b \to I(a)$, $a \to I(b)$ and $a \to O(z)$, i.e. $O(z) \subseteq O(a)$, $I(a) \subseteq I(z)$. Since $d^-(b) \geq 1$, $a \to I(b) \to b$. Since z is a king of $T - \{a, b\}$, $z \to O(z) \dashrightarrow I(z)$. Thus $a \to O(z) \dashrightarrow I(a)$ and $a \to z$ (see Figure 3). This shows that a is king of D, a contradiction.

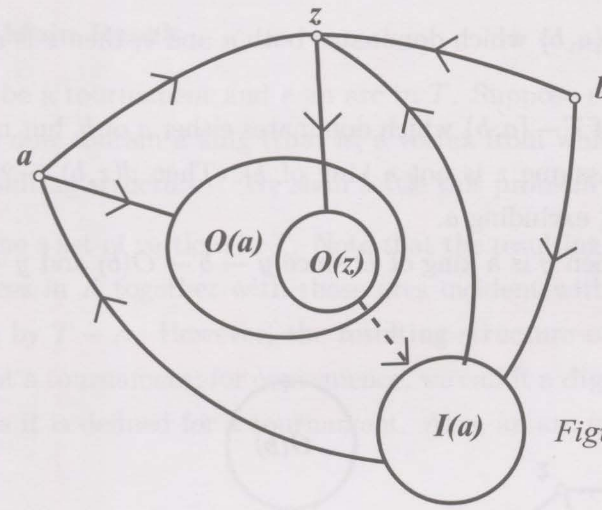

Figure 3

The proof of Theorem 4 is thus complete. □

Remark. Let e and f be two arcs in a tournament T. What are the conditions that should be imposed on e and f so that the digraph obtained by deleting them from T contains a king? We shall study this more complicated problem in another (forthcoming) note.

5. Acknowledgements

We would like to thank **Professor Koh Khee Meng** from the Department of Mathematics, National University of Singapore for his guidance throughout the entire course of Maths Research Programme 2005/06, which is organized by the **Department of Mathematics, Hwa Chong Institution (College)**. We would also like to acknowledge **Mr Andrew Yap** from the Department of Mathematics, Hwa Chong Institution (College) for his continuous supervision and support during the programme.

References

[1] G.Chartrand and P. Zhang, "*Introduction to Graph Theory*", McGraw-Hill International Edition, 2005.

[2] K. M. Koh, "*Digraphs*", lecture notes, National University of Singapore.

[3] H.G. Landau, *On dominance relations and the structure of animal societies III.* The condition for a score structure. ***Bill. Math. Biophys.***, 15(1953), 143 — 148.

[4] J.W. Moon, *Solution to problem 463.* ***Math. Mag.***, 35(1962), 189.

Hwa Chong Institution (College), 661 Bukit Timah Road, Singapore 269734