Kings in Tournaments (2)

by
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Abstract

In [5], we have characterized those arcs $e$ in a tournament $T$, such that the digraph $T - \{e\}$ obtained by deleting $e$ from $T$ contains a king. In this note, we characterize those pairs of arcs $\{e_1, e_2\}$, where $e_1 = (a, b)$ and $e_2 = (b, c)$ such that the digraph $T - \{e_1, e_2\}$ obtained by deleting these two arcs from $T$ contains a king.

1. Tournaments

A tournament is a non-empty finite set of vertices in which every two vertices are joined by one and only one arrow (such an arrow is also called an arc or a directed edge).

Let $T$ be a tournament and $x, y$ be two vertices in $T$. If there is an arrow from $x$ to $y$, we say that $x$ dominates $y$ or $y$ is dominated by $x$ (symbolically, $x \rightarrow y$). An arc from $x$ to $y$ is denoted by $(x, y)$. The number of vertices dominated by $x$ is the out-degree of $x$, and is denoted by $d^+(x)$. The number of vertices that dominate $x$ is the in-degree of $x$, denoted by $d^-(x)$. The set of vertices dominated by $x$ is the out-set of $x$, $O(x)$; and the remaining set of vertices that dominate $x$ is the in-set of $x$, $I(x)$.

Let $x$ be a vertex and $A$ be any set of vertices not containing $x$ in $T$. We write $A \rightarrow x$ to indicate that every vertex in $A$ dominates $x$; and $x \rightarrow A$ to indicate that $x$ dominates all the vertices in $A$. We write $A \Rightarrow x$ to indicate that at least one vertex in $A$ dominates $x$; and $x \Rightarrow A$ to indicate that $x$ dominates at least one vertex in $A$.

For any two vertices $x, y$ in $T$, the distance from $x$ to $y$, denoted by $d(x, y)$, is the minimum number of arrows one has to follow in order to travel from $x$ to $y$. Clearly, $d(x, y) = 1$ if $x$ dominates $y$; $d(x, y) \geq 2$ if $x$ does not dominate $y$. Also, we write $d(x, y) = \infty$ if $y$ is not reachable from $x$.

2. Kings in Tournaments

Let $T$ be a tournament with $n \geq 2$ vertices. A vertex $x$ in $T$ is called the emperor if $d(x, y) = 1$ for any other vertex $y$ in $T$; a vertex $x$ in $T$ is called a king if $d(x, y) \leq 2$ for any other vertex $y$ in $T$.

Studying dominance relations in certain animal societies, the mathematical biologist Landau proved in [3] the following result:
Theorem 1. In a tournament $T$, any vertex with the maximum score (out-degree) is always a king.

Moon, a Canadian mathematician, proved in [4] the following:

Theorem 2. In a tournament $T$, any non-emperor vertex $v$ (i.e. $v$ is dominated by some other vertex in $T$) is always dominated by a king.

As a direct consequence of Theorem 2, we have:

Corollary 3. No tournament contains exactly two kings.

Thus, any tournament either contains exactly one king (the emperor) or at least three kings.

Let $D$ be the resulting structure obtained from a tournament by deleting some arcs. A vertex $x$ in $D$ is called a king if $d(x,y) \leq 2$ for any other vertex $y$ in $D$. In [5], we have proved the following result:

Theorem 4. Let $T$ be a tournament with at least three vertices and $e = (a,b)$ an arc in $T$. Let $D = T - \{e\}$. Then $D$ contains at least one king if and only if $d^-(a) + d^-(b) \geq 1$ in $D$.

3. The Main Result

Now let $T$ be a tournament and $e_1, e_2$ be two arcs in $T$. Suppose $e_1$ and $e_2$ are deleted from $T$. Does the resulting structure still contain a king?

The objective of this note is to establish the following result.

Theorem 5. Let $T$ be a tournament with at least three vertices and $e_1 = (a,b), e_2 = (b,c)$ two arcs in $T$. Let $D = T - \{e_1, e_2\}$. Then $D$ contains at least one king if and only if $d^-(a) + d^-(b) \geq 1$ and $d^-(b) + d^-(c) \geq 1$ in $D$, not counting $(a,c)$ (i.e., $d^-(b) \geq 1$ OR $d^-(b) = 0, d^-(a) \geq 1$, and $d^-(c) \geq 1$ not counting $(a,c)$.)

Note: The dotted lines show that the arcs $(a,b)$ and $(b,c)$ are deleted. $L$ is the set of vertices excluding $a$, $b$ and $c$ in $D$. The dotted arrows indicate that the dominance relations are arbitrary and to be discussed.
Proof: [Necessity] Suppose on the contrary that in $D$, $d^-(a)+d^-(b) < 1$, i.e. $d^-(a)+d^-(b) = 0$, not counting $(a,c)$. Then $d^-(a) = 0$ and $d^-(b) = 0$. In this case, $d(x,b) = \infty$ for every $x$ in $D$ and $d(b,a) \geq 3$. Thus $D$ contains no kings. Similarly, if $d^-(b) + d^-(c) < 1$, then $D$ contains no kings either.

[Sufficiency] Case (1) $d^-(b) \geq 1$ in $D$.

Let $x$ be any vertex that dominates $b$, i.e. $x \rightarrow b$. If $x$ is the emperor of $B(= D - \{b\})$, then $x$ is the only king of $D$.

If $x$ is not the emperor of $B$, then $x$ is dominated by a king of $B$, by Theorem 2. Let this king be $z$. Clearly $d(z,b) = 2$, and thus $z$ is a king of $D$ (see Figure 2).

Case(2) $d^-(b) = 0$, $d^-(a) \geq 1$ and $d^-(c) \geq 1$ in $D$, not counting $(a,c)$.

Then $b \rightarrow L$, and $d(x,b) = \infty$ for any $x$ in $D$. Clearly, $b$ is the only king of $D$ since $b \rightarrow L \rightarrow a$ and $b \rightarrow L \rightarrow c$.

The proof of Theorem 5 is thus complete. \(\square\)

Now consider a more general problem. Let $T$ be a tournament with at least four vertices and $e_1 = (a,b), e_2 = (c,d)$, where $b$ and $c$ may not be the same, be two arcs in $T$. Let $D = T - \{e_1, e_2\}$. Are similar conditions (i.e. $d^-(a)+d^-(b) \geq 1$ and $d^-(c)+d^-(d) \geq 1$ in $D$) sufficient to ensure the existence of a king of $D$? We can easily find one counter example: Suppose that in $D$, $d^-(a) = d^-(b) = d^-(c) = d^-(d) = 1$, and $a \rightarrow L, b \rightarrow L, c \rightarrow L, d \rightarrow L$ (see Figure 3).
In this case, \( D \) does not contain a king.

Thus, what additional conditions should be imposed so that \( D = T - \{e_1, e_2\} \) always contains a king? This problem remains open.

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References


