Selected Problems from

Moscow and St. Petersburg Mathematics Olympiads

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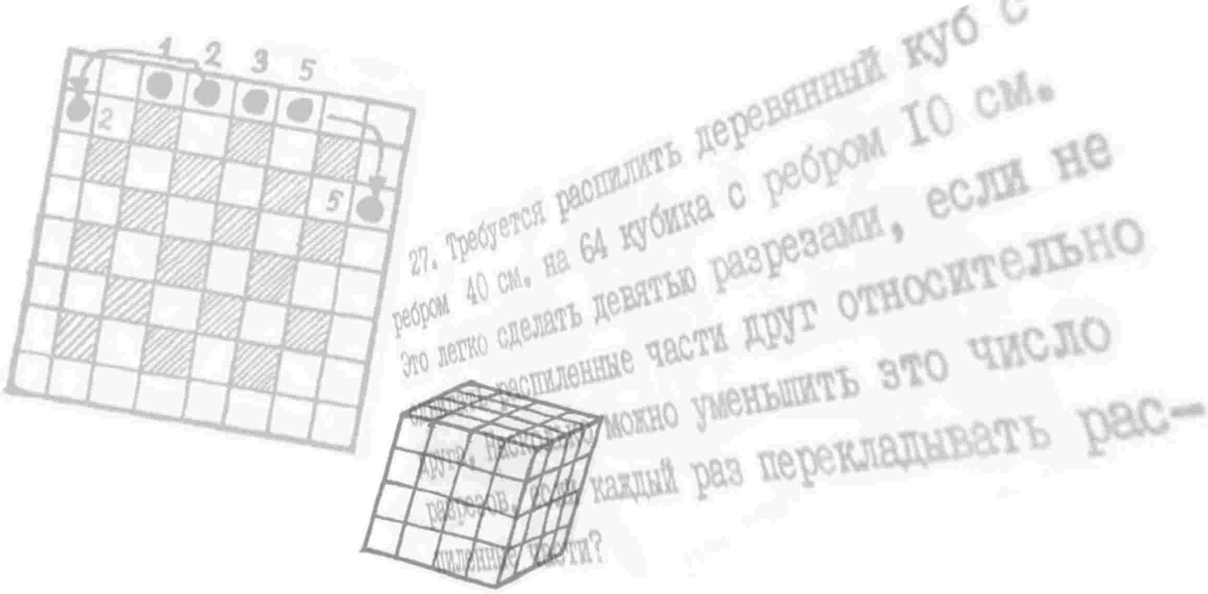
Introduction

Moscow and St. Petersburg are the two biggest Russian cities, with the history of mathematics olympiads going back to 1930s. Both cities set six different test papers for grades 6 to 11 (roughly corresponding to ages 12 to 17 years old respectively).

Nowadays, the olympiad in Moscow is a written exam. Each question is graded by a large team of judges (mostly students and professors of Moscow State University) according to the marking scheme they develop. Complete solutions get full score. If there are flaws, then the judges penalize the participant by deducting some marks. If a contestant didn't solve a problem but had some interesting ideas, then he or she is rewarded with some bonus points. The winner is then the participant whose total score is the highest.

The St. Petersburg olympiad is oral. Each participant is supposed to claim that a question is solved and explain the solution to a judge. Up to three attempts per question are allowed. Thus there is no such thing as intermediate score for interesting ideas or incomplete solutions, and the winner is the participant with the most questions solved and successfully presented to the judges.

Traditionally, areas of geometry and combinatorics are considered main strengths of Russian mathematics olympiads. In this article, we offer a selection of geometry and combinatorics problems of the 2012 Moscow and St. Petersburg olympiads, one from each exam paper.



Problems

St. Petersburg, 6th grade Each of the 25 girls has friends among some of the 25 boys. A girl can start a new life — break up with all her boy-friends and befriend every boy who was not her friend previously. Prove that it is always possible for some of the girls to start a new life so that, as a result, there exist three boys having the same or almost the same number of girl-friends (numbers are considered "almost the same" if the differences between them do not exceed 1).

St. Petersburg, 7th grade Is it possible to arrange numbers $99, 100, \ldots, 200$ in a row so that in every pair of neighbour numbers, one of them is either twice as large as, or by 2 larger than the other one?

St. Petersburg, 8th grade A liar always lies and a knight always tells the truth. Every participant of the conference 'Truth and Lie in Modern Society' was either a knight or a liar. All the conference participants were seated at a round table. Then, each of them turned to each of his neighbours and said either 'You're a knight' or 'You're a liar'. A reporter then asked each of the participants two questions 'Were you called a liar by your left neighbour?' and 'Were you called a liar by your right neighbour?' Precisely 100 of the answers were 'yes'. What is the minimal possible number of liars among the conference participants?

St. Petersburg, 9th grade Let $ABCD$ be an inscribed quadrilateral. The bisector of the angle between its two diagonals intersect the sides AB and CD at the points X and Y respectively. It is known that the midpoint of the side AD is equidistant from X and Y. Prove that the midpoint of the side BC is also equidistant from X and Y.

St. Petersburg, 10th grade 100 unit line segments are given in the first coordinate quadrant. No two of them intersect each other and each one is parallel to one of the coordinate axes. The line segments are double-sided mirrors. The mirrors reflect a ray of light, making the angle of incidence equal the angle of reflection, except for the endpoints of each line segment, which the light passes without changing its direction. The source located within the unit distance from the origin sends a beam of light in the direction parallel to the line $y = x$ such that x and y are increasing. Prove that it's possible to find an appropriate position of the source so that the ray of light reflects from the mirrors at most 150 times.

St. Petersburg, 11th grade The base of a pyramid $SABCD$ is a convex quadrilateral $ABCD$ such that $BC \cdot AD = BD \cdot AC$. It is known that $\angle ADS = \angle BDS$ and that $\angle ACS = \angle BCS$. Prove that the plane SAB is perpendicular to the plane $ABCD$.

Moscow, 6th grade Cut the frame shown in Figure 1 into 16 equal pieces.

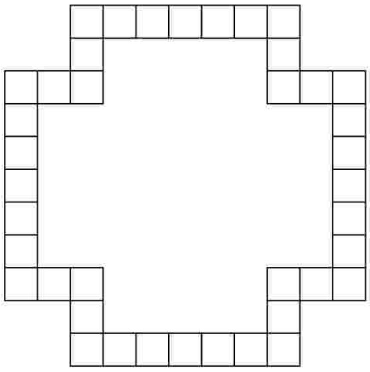


Figure 1: Moscow, 6th grade

Moscow, 7th grade The 3×3 square is filled with numbers $1, \ldots, 9$ as shown in Figure 2 on the left. It is allowed to move in the square, each time passing from a cell to one of the neighbour cells so that no cell is visited twice. For instance, Alice's path is the one shown in Figure 2 on the right. Alice has written down the digits on her way through the square and obtained the number 73948165. Find another path such that the number you will produce by writing down the digits along the way is as large as possible.

1	8	4
6	3	9
5	7	2

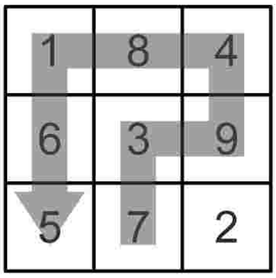


Figure 2: Moscow, 7th grade

Moscow, 8th grade Consider 100 points on the plane such that no three of them are collinear. Is it always possible to split them into pairs and join the points of every pair with a line segment such that every two of these line segments intersect?

Moscow, 9th grade Let $ABCD$ be a parallelogram and let BH be the perpendicular to the side AD. Further, let $M \in BH$ be the point equidistant from C and D. Finally, let K be the midpoint of the side AB. Prove that $\angle MKD$ is a right angle.

Moscow, 10th grade Entries of an $n \times n$ table are signs $+$ and $-$. At every step, you are allowed to reverse all signs in a particular row or a particular column. It is known that after several such operations you can get the table all of whose entries are $+$'s. Prove that you can do it in at most n steps.

Moscow, 11th grade Let X be an infinite collection of rectangles satisfying the following property:

- given any positive number S, there exist rectangles in X whose sum of areas exceeds S.

Answer the following two questions:

(a) Is it true that it is always possible to cover the whole plane with rectangles from the collection X if different rectangles are allowed to overlap?

(b) Additionally, suppose that all rectangles in X are squares. Then, is it true that it is always possible to cover the whole plane with rectangles from X if they are allowed to overlap?

If you manage to solve any of the problems above, you are welcome to send your solution to Dr Duzhin at fduzhin@gmail.com or post it on Singapore Mathematical Society's Facebook: http://www.facebook.com/SingaporeMathSoc.

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