# On the Regiomontanus Angle Maximization Problem

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**Abstract**

The Regiomontanus angle maximization problem is a famous historical problem posed by Johannes Müller (alias Regiomontanus) in the 15th century. We present a simple geometric approach to solve this problem without having to use calculus or the AM-GM inequality.

## 1. Introduction

In 1471, the German mathematician Johannes Müller (1436-1476), alias Regiomontanus, posed the following problem in a letter to Christian Roder, who was a professor at the university of Erfurt: "At what point on the ground does a perpendicularly suspended rod appear the largest (i.e. subtends the greatest visual angle)?" (see [4]). This problem can be illustrated in Fig. 1 below, where AB represents the rod, OA=$a$, OB=$b$, OP=$x$ and P is the point to be located on the ground so that ∠APB (θ) is maximum.

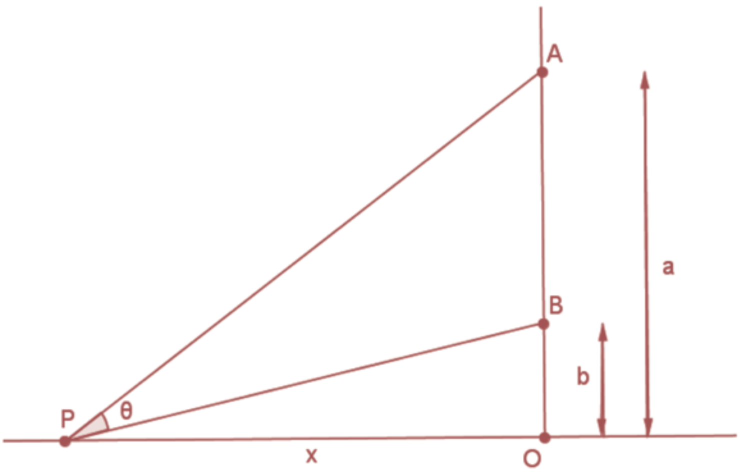


Figure 1

Various versions of this problem are mentioned in [1, 2, 3, 6, 7, 8] and the solutions described are very similar to each other. For instance, Maor (1998) used a trigonometry formula to obtain:

$$\cot\theta = \cot(\alpha-\beta) = \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha} = \frac{(x/a)(x/b)+1}{x/b-x/a} = \frac{x}{a-b}+\frac{ab}{(a-b)x},$$

where $\alpha = \angle APO$ and $\beta = \angle BPO$. Then, we can use standard calculus technique to determine the value of $x$ such that cot $\theta$ is minimum (or $\theta$ is maximum). Alternatively, we can solve the problem by using the inequality of arithmetic mean and geometric mean (AM ≥ GM) (see [4]). By putting $u = x/(a-b)$ and $v = ab/(a-b)x$, the above equation becomes

$$\cot\theta = u+v \geq 2\sqrt{uv} = 2\sqrt{ab}/(a-b).$$

We can see that cot $\theta$ is minimum when $u$=$v$ or $x=\sqrt{ab}$. Is there a simple geometric approach to solve this problem without using calculus or the AM-GM inequality? In this article, we present such an approach, which requires basic Euclidean Geometry knowledge only.

## 2. Basic results in Euclidean Geometry

The following theorems of Euclidean Geometry (see [5]) are often covered in the high school mathematics curriculum, namely:

Theorem 1: Three non-collinear points uniquely define a circle.

Theorem 2: Perpendicular bisectors of chords of a circle pass through the centre of the circle.

Theorem 3: The angle at the centre is twice the angle at the circumference if they subtend the same chord in a circle.

Theorem 4: The exterior angle of a triangle is greater than each of the remote interior angles.

We will describe how to use these results to solve the Regiomontanus angle maximization problem in the next section.

## 3. A simple geometric approach

In Fig. 2, the points A, B are fixed and P is a point on the ground, whose position is to be located such that ∠APB (θ) is maximum. By Theorem 1, there exists a circle which passes through these three points. Let C, M be the mid-point of AB and the centre of the circle respectively. By Theorem 2 and Theorem 3, M lies on the perpendicular bisector of AB and θ is equal to half of ∠AMB (i.e. θ=∠AMC). Thus, the Regiomontanus problem is equivalent to the problem of locating the position of M along the perpendicular bisector of AB such that ∠AMC is maximal and the circle intersects the horizontal line on the ground.

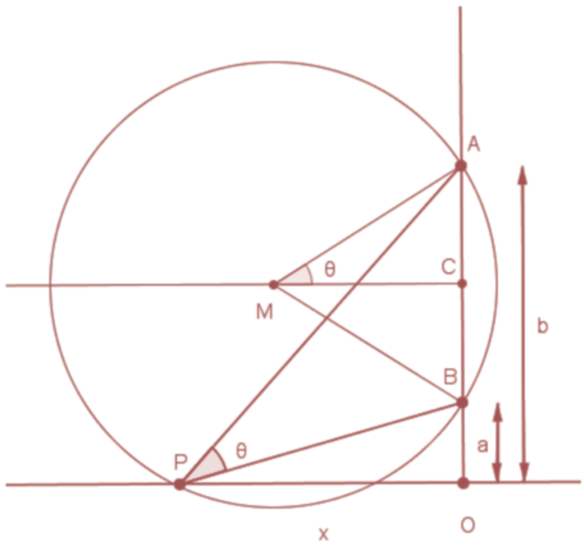


Figure 2

Let $r$ be the radius of the circle. Now, if the point M moves along the perpendicular bisector of AB towards the point C, then $r$ decreases as illustrated in Figure 3, where the new positions of M, P, θ are denoted by M', P', θ', respectively. Also, according to Theorem 4, we have $\theta < \theta'$, since the latter one is an exterior angle of ΔAMM'.

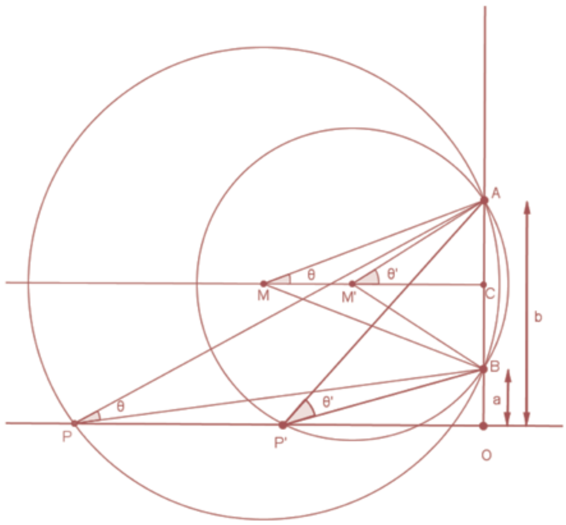


Figure 3

Now, there are three possible cases to be examined, namely:

Case I: If $r >$ OC, then the circle intersects the horizontal line on the ground at two points, as shown in Figure 2.

Case II: If $r =$ OC, then the circle intersects the horizontal line on the ground at one point only, which means OP is tangent to the circle at P (see Fig. 4).

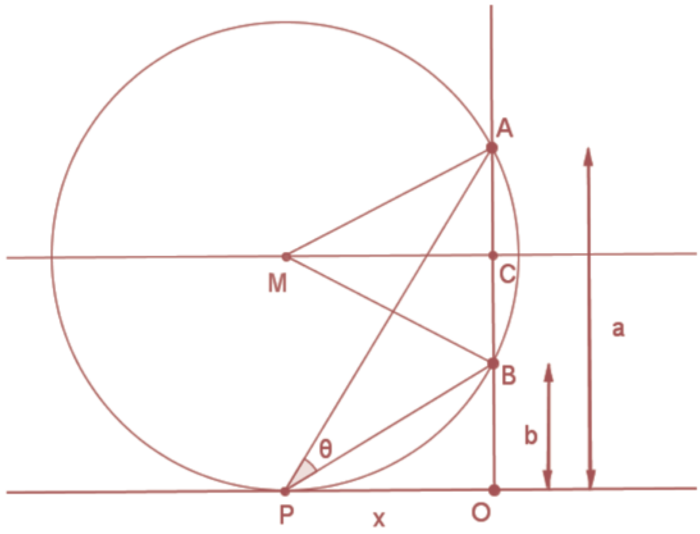


Figure 4

Since M is now closer to C, so ∠AMC is greater than the corresponding angles in Case I, as explained and illustrated in Fig. 3 above. By using the Pythagoras Theorem,

$$r^2 = AC^2 + MC^2.$$

So, $(b+\frac{a-b}{2})^2 = (\frac{a-b}{2})^2 + x^2$. Hence, $x^2 = ab$, which implies $x = \sqrt{ab}$.

Case III: If $r <$ OC, then the circle does not intersect the horizontal line on the ground. There will be no solution in this case.

In brief, the solution to the Regiomontanus angle maximization problem exists in Case II, when OP is tangent to the circle passing through A, B, P and the horizontal distance of P from OA is equal to $\sqrt{ab}$.

### 4. Hints on geometric constructions by means of dynamic geometry software

In this article, we have introduced a simple geometric approach to solve the Regiomontanus angle maximization problem, without having to use calculus or the AM-GM inequality. This approach can provide a stepwise construction procedure for locating the centre (M) of the circle and the position of P by means of common dynamic geometry software, namely:

1) First, draw a perpendicular bisector ( $l$ ) to AB. Label the midpoint of AB as C.

2) Next, use A as the centre and OC as the radius to draw an arc to intersect $l$ to locate M.

3) Last, use M as the centre and OC as the radius to intersect the horizontal line through O to locate the point P.

### References

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