Generalized Mersenne Numbers and Primitive Pythagorean Triples

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Abstract

The famous Mersenne numbers $M_n = 2^n - 1$ are generalized and questions regarding whether these numbers can be connected to primitive Pythagorean triples are raised and answered.

The Mersenne numbers, named after the priest Marin Mersenne (1588-1648) who was a correspondent of Fermat, are defined by $M_n = 2^n - 1$ for all positive integers n. These numbers are justly famous because the largest known prime at any time is usually a Mersenne number. In this article we generalize them to numbers of the form $M_{k,s} = 2^k - s^2$ for positive integers k and odd positive integers s. We may refer to these numbers as Mersenne numbers of order s, so that a Mersenne number of order 1 is the usual Mersenne number described above.

A primitive Pythagorean triple (PPT) is a triple of positive integers (a,b,c) satisfying $a^2 + b^2 = c^2$ as well as $\gcd(a,b) = 1$. It has been known since the time of the ancient Greeks (Euclid's Elements, Book X, Prop. 29) that a PPT is completely determined by parameters m and n that are unequal positive integers of opposite parity such that $\gcd(m,n) = 1$. Indeed, we have for any PPT (a,b,c) that $a = 2mn, b = m^2 - n^2, c = m^2 + n^2$ for appropriate m and n, provided $m > n$. Without loss of generality we are assuming that a is the even leg and b is the odd leg.

We can provide two infinite families of PPTs whose odd legs are Mersenne numbers as follows. Set $m = 2^k$ and $n = 1$ to get $b = 2^{2k} - 1 = M_{2k}$. Alternatively, set $m = 2^k$ and $n = 2^k - 1$ to get $b = 2^{k+1} - 1 = M_{k+1}$.

We can find infinitely many PPTs whose sum of legs $a + b$ is a Mersenne number. Set $m = 2^k$ and $n = 2^k - 1$ so that $a + b = 2^{k+1}(2^k - 1) + 2^{2k} - (2^k - 1)^2 = 2^{2k+1} - 1 = M_{2k+1}$.

The next two results generalize the previous ones.

Fix s^2 to be the square of an odd positive integer. We can prove that there are infinitely many PPTs with odd leg of the form $2^t - s^2 = M_{t,s}$. Simply let $m = 2^k$ and $n = s$ with k chosen so that $2^k > s$. Then $(2^{2k} - s^2, s \cdot 2^{k+1}, 2^{2k} + s^2)$ is a PPT with odd leg $M_{2k,s}$. This is easily verified.

Let s be a fixed odd integer. Are there infinitely many PPTs whose sum of legs is a generalized Mersenne number of order s ? The affirmative answer is obtained as follows. Let $m = 2^k$ and $n = 2^k - s$. Then $a + b = 2^{k+1}(2^k - s) + 2^{2k} - (2^k - s)^2 = 2^{2k+1} - s^2 = M_{2k+1,s}$.

Next we show that the product bc of the odd leg and hypotenuse of a PPT can be a Mersenne number of order s^2 infinitely often. To see this, for fixed s, let $m = 2^k$ and $n = s$ with $2^k > s$ to get $bc = (2^{2k} - s^2)(2^{2k} + s^2) = 2^{4k} - s^4 = M_{4k,s^2}$.

Now, suggested by our investigations using Mersenne numbers of order s, we stray slightly and consider numbers of the form $N_{k,s} = 2^k + s^2$ for fixed odd integer s and positive integers k. We can show that, given any such s, there is at least one PPT with odd leg $N_{k,s}$ for some k. Put $m = \dfrac{3+s^2}{2}$ and $n = \dfrac{1+s^2}{2}$. These are both integers and, because they are consecutive integers, they are of opposite parity and relatively prime. Thus they generate a PPT with odd leg

$$m^2 - n^2 = \left(\frac{3+s^2}{2}\right)^2 + \left(\frac{1+s^2}{2}\right)^2 = \frac{8+4s^2}{4} = 2 + s^2 = N_{1,s} \text{ as desired.}$$

Finally, we leave an exercise for the reader. Can both the odd leg and the hypotenuse of a PPT be Mersenne numbers?

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