Conway's Game of Life

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Abstract

Conway's Game of Life is a cellular automaton devised by the British mathematician John Conway. As the original Game of Life is based on a plane of tessellated squares and lacks analysis on other tessellations, this project aimed to investigate a variation of the Life using hexagonal tessellation and hence determine the fractal patterns in chaotic growth. Through writing computer programs and running simulations for both the original Game of Life and the Hex Life, the best set of rules was derived to ensure continuous growth of the Hex Life. Both the original Game of Life and the Hex Life were also proven to be fractals mathematically, with self-similar patterns.

Introduction

Conway's Game of Life is a cellular automaton devised by the British mathematician John Horton Conway in 1970. The evolution of the game is determined by its initial position.

The universe of the Game of Life is an infinite two-dimensional orthogonal grid of square cells, each of which is either alive or dead. Every cell reacts with the 8 surrounding cells, or the Moore Neighborhood, as shown in figure 1.

$(i-1,j-1)$	$(i,j-1)$	$(i+1,j-1)$
$(i-1,j)$	(i,j)	$(i+1,j)$
$(i-1,j+1)$	$(i,j+1)$	$(i+1,j+1)$

Figure 1: The Moore Neighborhood consists of the 8 surrounding cells

Before going on to the rules of the game, our group would like to introduce the set of notation that we have used in the following figures. Firstly, a live cell is represented by a black cell, while a dead cell is represented by a white cell. Next, the numbers within each cell represents the number of life cells in the Moore Neighborhood.

The rules of the game are that:

1. Any live cells with fewer than two live neighbors dies by under-population, as shown in figure 2.

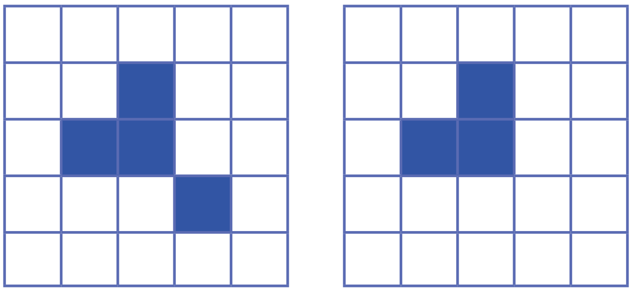

Figure 2: The given example ignores all other rules

2. Any live cells with two or three live neighbors lives on to the next generation. As shown in figure 3

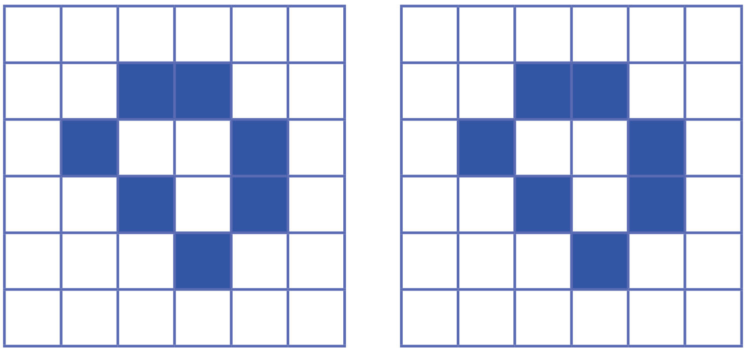

Figure 3: The given example ignores all other rules

Any live cell with more than three live neighbors dies by overcrowding

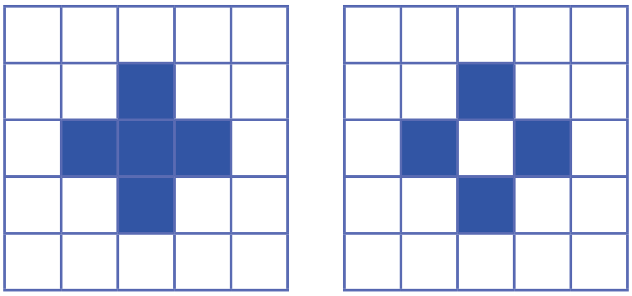

Figure 4: The given example ignores all other rules

3. Any dead cell with exactly three live neighbors becomes a live cell by reproduction.

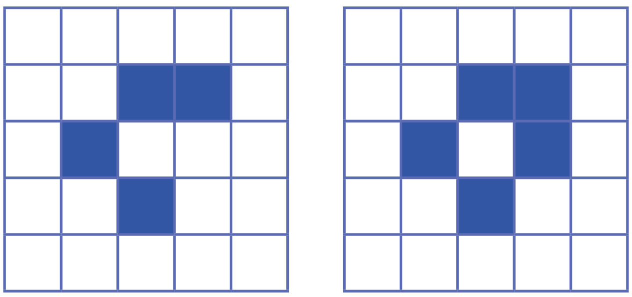

Figure 5: The given example ignores all other rules

These rules would be applied in each generation until the cells die out, or becomes stable. Stable life comprises of two different components. Firstly, an oscillator is a pattern that returns to its original state in the same orientation and position after a finite number of generations. Oscillators have different degrees of oscillation. An oscillator with x degree of oscillation will require x number of generations before returning to its original state. The following figures are examples of oscillators.

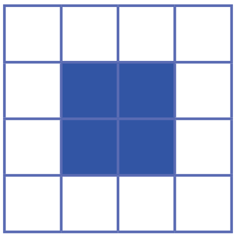

Figure 6: The above is an oscillator with 1 degree of oscillation

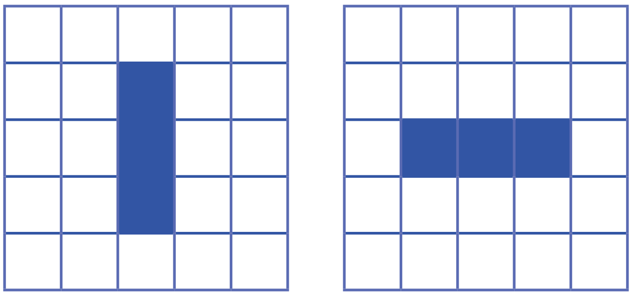

Figure 7: The above is an oscillator with 2 degrees of oscillation. It is also commonly known as "blinker"

Next, spaceships are patterns that reappear after a certain number of generations in the same orientation but in a different position, as shown in figure 8.

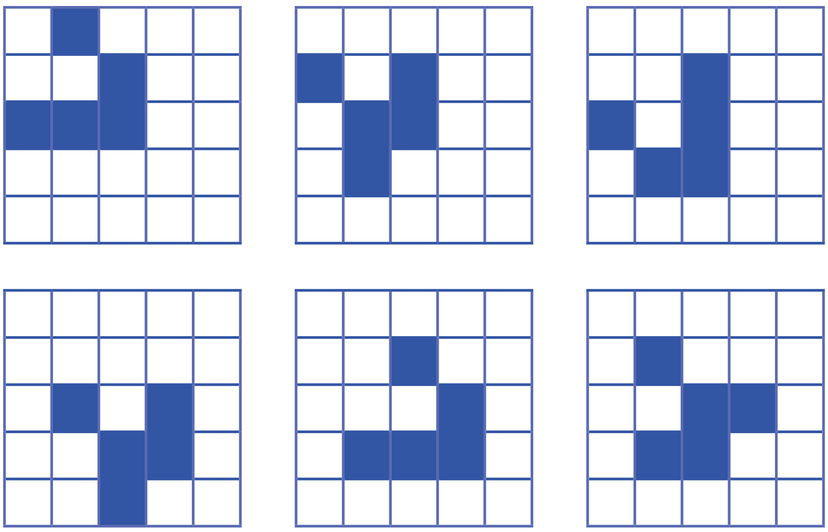

Figure 8: This spaceship is often known as the "glider"

Conway's Game of Life is mainly focused around square tessellations, perhaps because Conway felt that other tessellations of shapes would not be able to sustain life as there are too little neighboring cells to interact with. Therefore, our group wants to experiment with hexagonal tessellations.
One of the prerequisites for Conway's Game of Life is that the game must satisfy class 4 complexity. As mentioned by Ilachinski (2001), in class 4 complexity, nearly all initial patterns evolve into structures that interact in complex ways, with formation of local structures that are able to survive for many generations. With this in mind, our project seeks to identify a rule for Hexagonal-Life, such that class 4 complexity can be achieved.

Terminology

- **Hex-Life:** Revised Game of Life in Hexagonal Tessellation.
- **Moore Neighborhood:** The number of cells that surround a central cell. (In the conventional Conway's game of life, each cell has 8 neighboring cells, whereas in Hex-Life, each cell only has 6 cells.)
- **B*x*:** *x* represents the number of cells required in the Moore Neighborhood that will result in birth of the central cell. For example, in the original Game of Life, we have **B3**, meaning that birth will occur when there are exactly 3 cells.
- **S*yz*:** *y* and *z* represent the number of cells required in the Moore Neighborhood that will result in survival of the central cell. For example, in the original Game of Life, we have **S23**, meaning that a live cell will survive when there are 2 or 3 cells.
- **First-Order Neighbors:** The cells directly adjacent to a central cell.
- **Second-Order Neighbors:** The cells that are adjacent to two First-Order Neighbors.
- **Bounding Box:** The smallest rectangle that can cover all alive cells (in the Original Game of Life) or the smallest parallelogram that can cover all alive cells (in Hex-Life).

Literature Review

Bays (1987) attempted to generalize Conway's Game of Life into 3 dimensions. His work helped to give us insight on possibilities to expand Conway's Game of Life, and also helped us to identify what should be done in our research. The largest flaw in Bays' work is the lack of connection between his works and that of Conway's. There was no cross referencing between his work and Conway's work. Due to the large number of possibilities of rule sets that Conway's Game of Life can have, our group found it important that we use the original Game of Life as a benchmark for our own rule-set in Hex-Life.

Apart from Bays' paper, a program entitled "Hexlife" was also found online. Since there are only 6 cells surrounding each cell, the program tries to compensate for this by extending the neighbourhood to those which are touching two of the adjacent cells, as shown in the rules page of the site. Trying to compensate for exponential growth, which would result if one weighed all those cells equally and used a simple rule, David G. Ballinger gave the immediately adjacent cells ("first-order neighbours") a weight of 1, and the other cells which are counted ("second-order neighbours") that of 0.3, so the total count is 9.8, close to 10. By accounting for differences in the two versions of Life, the creator of Hex-Life came up with the range for a cell surviving and being born, 2.0 to 3.3 and 2.6 to 3.3 respectively. However, our group feels that the "second-order neighbors" is unnecessary. Our group believes that hexagonal life can reach class 4 complexity without considering 12 cells, but the 6 cells in the Moore Neighborhood.

Research Questions

1. By comparing number of live cells and bounding box of the expansion after 20 generations, we aim to find the situation of Hex-Life that is closest to Conway's Game of Life and therefore, deduce the best set of rules to ensure continuous growth.

2. We would also like to investigate fractal properties in Hex-life using the Minkowski-Bouligand dimension.

RESEARCH PROBLEM I

Proposed Methodology

1. Write a simple program to help make the simulations easier.
2. Record down the bounding box and number of life cells in all possible permutations of cells in a 5 by 5 box in the actual Conway's Game of Life after 20 generations.
3. Record down the bounding box and number of life cells in all possible permutation of cells in a "honeycomb" like grid (7 cells) using different rules of Hex-Life after 20 generations.
4. Using Excel, compare and find the hex-life rule that is closest to the actual Conway's Game of Life.

Results

We determined the live cells and bounding box by finding the ratio of the two terms after 20 generations. The initial conditions when running simulations for the original Game of Life would be all polyminos within a square with length and breadth of 3 units, whereas, the initial conditions used in testing for Hex-Life would be all the polyminos within a hexagon with 7 units. The bounding box is the smallest square and the smallest parallelogram that can cover all the live cells in Conway's Game of Life and Hex-Life respectively.

Number of cells needed for Birth in Hex-life	Similarity to Conway's Game of Life
B1	The birth rate is too high, resulting in exponential growth
B2	The results are slightly deviated with the original Game of Life
B3	The birth rate is too low, and most of the patterns die out.
B4	The birth rate is too low, and most of the patterns die out.
B5	The birth rate too low and almost all initial conditions die out immediately.
B6	The birth rate too low almost all initial conditions die out immediately.

The following figures are a scatter plot of the results from B1 and B2.

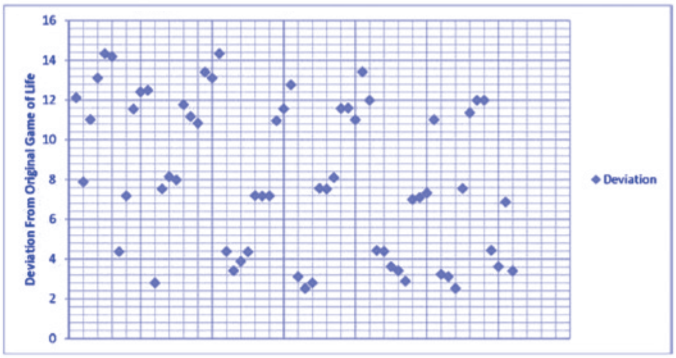

Figure 9: The above graph is the scatter plot for B1

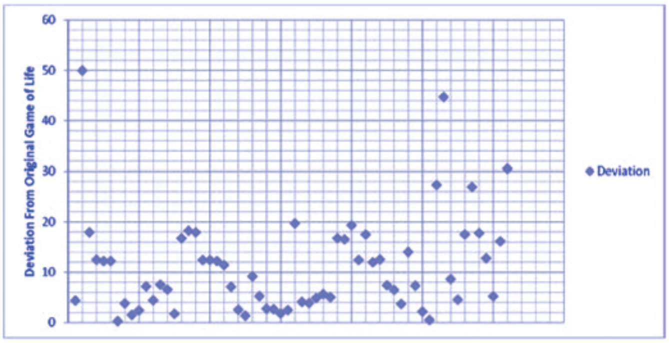

Figure 10: The above graph is the scatter plot for B2

After running tests on the original Conway's Game of Life, the expected live cells is 27.77% of the smallest square that can contain all the cells. The total amount of live cells in the bounding box is 1874 live cells in a bounding box of 6749 cells. Amongst all the rule-sets that we have tested, the rule-set B2S12, meaning that any dead cell will live if there are 2 neighboring cells, and each live cell will remain alive when there are 1 or 2 live cells provided us with a bounding-box to live cell ratio of 28.04, only 0.27 more than the original Game of Life.

B2S12 in Hex-life	Conway's Game of Life
28.04	27.77

Program Logic

In the program, each cell is represented by a 2-dimensional array, with the x-coordinate and the y-coordinate, as shown in figure 11. Each cell is assigned the values 0 or 1, with 0 representing a dead cell and 1 representing a live cell. Upon assigning an initial condition, the program will check all the surrounding cells to determine the position of the next central cell.

1,1	2,1	3,1	4,1	5,1	6,1	7,1	8,1	9,1	10,1
1,2	2,2	3,2	4,2	5,2	6,2	7,2	8,2	9,2	10,2
1,3	2,3	3,3	4,3	5,3	6,3	7,3	8,3	9,3	10,3
1,4	2,4	3,4	4,4	5,4	6,4	7,4	8,4	9,4	10,4

Figure 11: Data values for each cell in the original Game of Life

The concept in Hex-Life is very similar, except, every other row of cells is shifted ½ a cell to the right.

<table>
<tr><td colspan="2">1,1</td><td colspan="2">2,1</td><td colspan="2">3,1</td><td colspan="2">4,1</td><td colspan="2">5,1</td></tr>
<tr><td></td><td colspan="2">2,2</td><td colspan="2">3,2</td><td colspan="2">4,2</td><td colspan="2">5,2</td><td></td></tr>
<tr><td colspan="2">1,3</td><td colspan="2">2,3</td><td colspan="2">3,3</td><td colspan="2">4,3</td><td colspan="2">5,3</td></tr>
<tr><td></td><td colspan="2">2,4</td><td colspan="2">3,4</td><td colspan="2">4,4</td><td colspan="2">5,4</td><td></td></tr>
</table>

Figure 12: Data values for each cell in Hex-Life

Fractal Dimension in Hex-Life

Upon running many simulations for Hex-Life, we discovered that the Game of Life may contain fractal-like properties. There is a certain level of self-similarity in the expansions, with repeated clusters of cells in certain areas. As such, our group decided to try and investigate the possibility that Hex-Life could result in fractals.

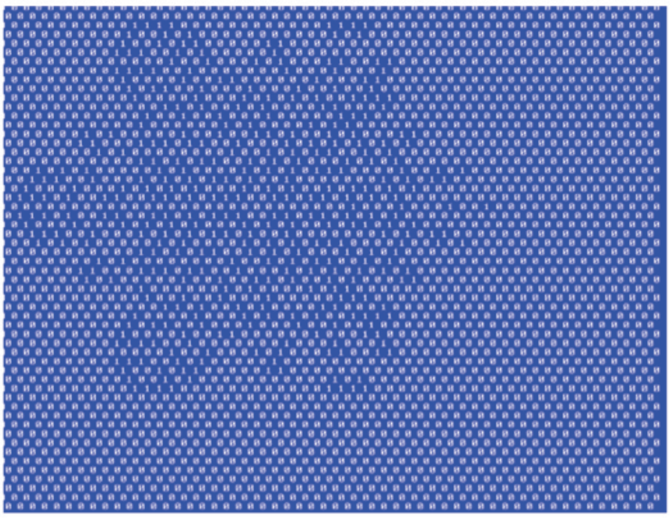

Figure 13: Small congregations of hexagon-like structures can be seen. This self-similarity is characteristic of a fractal.

RESEARCH PROBLEM II

Proposed Methodology

1. Research on the Hausdorff-Besicovitch Dimension in determining fractals.
2. Research on the Minkowski-Bouligand Dimension in determining fractals.
3. Evaluate between the methods.
4. Rewrite the program to calculate the possibility of fractal growth in chaotic growth.
5. Using Excel to evaluate and compile the results.

Definition of A Fractal

A fractal can be defined in terms of the Hausdorff-Besicovitch dimension or the Minkowski-Bouligand dimension. A set is called a fractal if its Hausdorff-Besicovitch dimension strictly exceeds the topological dimension. Due to the complexity of the Hausdorff dimension, our group decided to use the Minkowski-Bouligand dimension instead.
The Minkowski-Bouligand dimension is given by

$$-\lim_{\varepsilon \to 0} \frac{\log M_\varepsilon}{\log \varepsilon},$$

where M_ε represents mass and ε represents scale. More precisely, given a pattern, any fixed unit shape with side ε is tessellated across. The mass M_ε is the number of non-empty units. In any set, when the value of the Minkowski-Bouligand Dimension is non-integer, the pattern would be classified as a fractal.

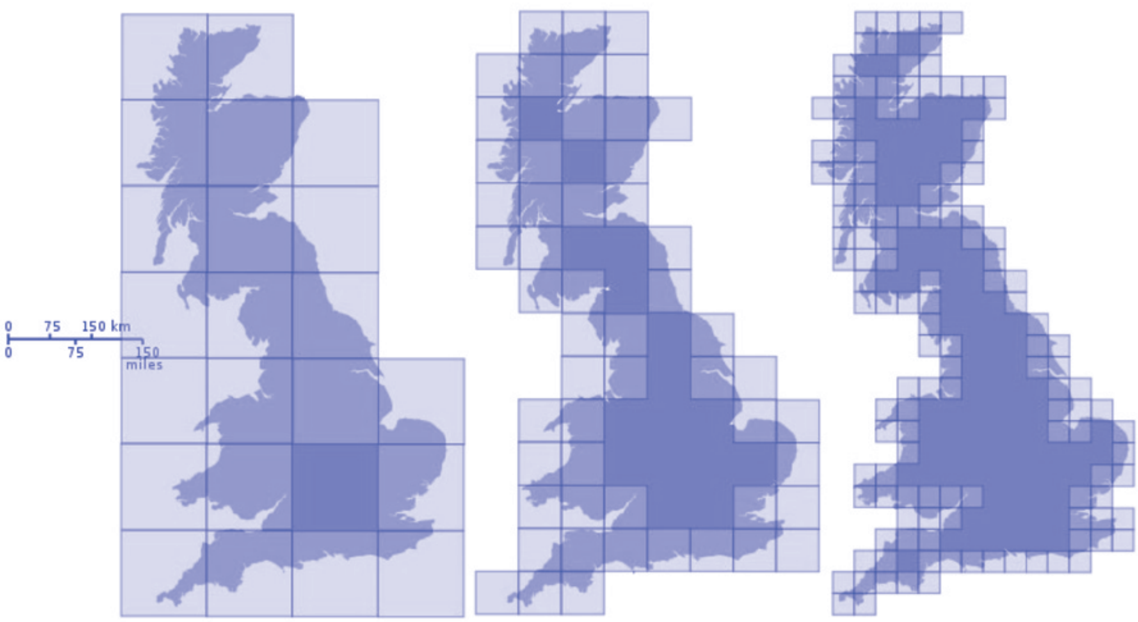

Figure 14: Estimation of the Minkowski-Bouligand dimension of the coast of Great Britain

(Source: http://en.wikipedia.org/wiki/Minkowski%E2%80%93Bouligand_dimension)

Results

Upon running the program for 1 billion generations, with 262144 cells and the initial condition of 10% randomly filled cells, our group came up with the fractal dimensions for both the original game of life and Hex-Life (B1S12 based on research problem 1).

The original game of life has a fractal dimension of 1.6622, while Hex-Life has a fractal dimension of 1.6625, meaning that they are mathematically fractals.

Our group also decided to extend the investigation of fractal dimension to other possible rule-sets of Hex-Life. Based on research problem 1, our group came up with 16 other rule-sets that deviated least from the original Game of Life.

Rule Set	Deviation	Fractal Dimension
B1S23	2.8	1.6639
B1S235	2.52	1.6621
B1S236	2.8	1.6637
B1S1256	2.89	1.6621
B1S2356	2.52	1.6631
B2S14	1.5	1.6639
B2S15	2.4	1.6622
B2S26	1.73	1.6643
B2S125	2.55	1.6619
B2S126	1.35	1.6627
B2S136	2.75	1.6639
B2S145	2.66	1.6625
B2S146	1.79	1.6634
B2S156	2.46	1.6635
B2S1356	2.17	1.6629
B2S1456	0.5	1.6623

All of the values for fractal dimension given by these rule-sets have a difference of at most 0.003, showing us that fractal properties do indeed exist in Hex-Life.

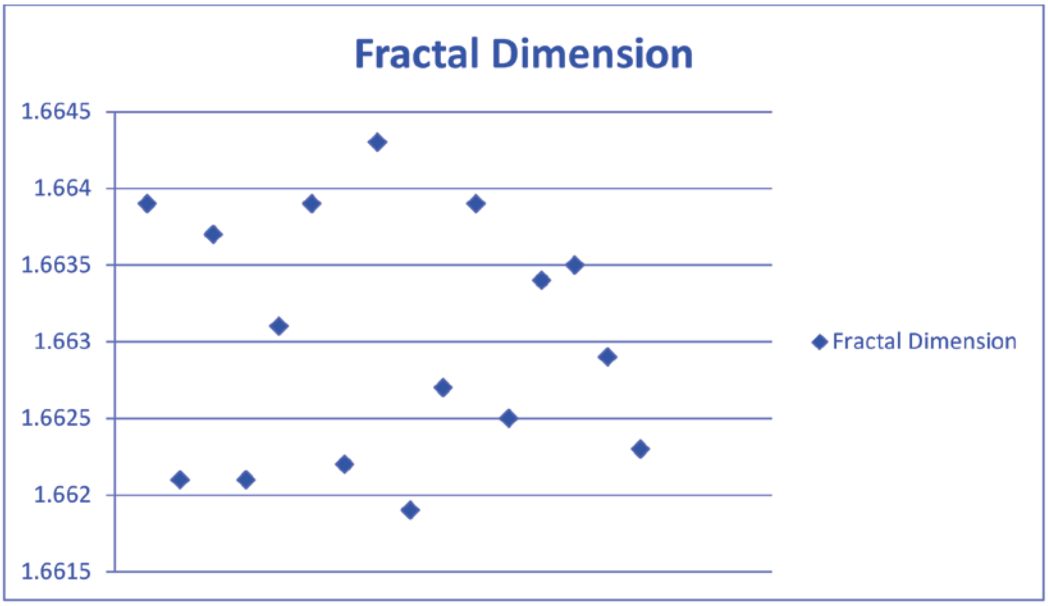

Figure 15: The above scatter plot is the fractal dimension of the chosen rule-sets

Conclusions

In summary, we found that the rule-set B2S12 is closest to the original Game of Life. We also discovered that Hex-Life will generate fractal-like patterns.

However, our project has some limitations, mainly

1. Sample Statistics is too small. Due to the lack of computational power, our group had to limit the sample size to a 5 by 5 box for the original Game of Life and a 7 cell honeycomb for Hex-Life.
2. The difference in some rule-sets are very small, therefore, the determining of the most ideal rule-set can be difficult
3. The initial position of the program was chosen randomly, meaning that the results may vary.
4. The size of the simulation was limited. After 1 billion generations, the growth of the cells will definitely have exceeded the limit that our program can hold. Therefore, the statistics that we have obtained is only a small portion of the actual position of the cells.

References

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