Domination in Digraphs

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Objectives

- Present some fundamental results on “dominating sets”
- Learn how to generalize or extend existing results
- Cultivate the Habit of Problem-Posing — the first step towards doing research
**Digraph** (Directed Graph)

**Digraph**: \( D = (V, A) \)

- \( V = \text{vertex set of } D \) = \{u, v, w, x, y, z\}
- \( A = \text{arc set of } D \) = \{xu, xv, uw, wu, \ldots, yz\}
• Domination
  \( u \) dominates \( x \)

• Degrees
  \( od(v) = \text{outdegree of } v = 3 \)
  \( id(v) = \text{indegree of } v = 2 \)

• Distance
  \( d(a, z) = \text{distance from } a \text{ to } z \)
  \( = \text{the min. no. of arcs traversed from } a \text{ to } z \)
  \( = 5 \)

Note \( d(z, a) = 3 \)
Dominating Vertices (Kings)

A vertex $w$ in $D$ is an $r$-king if $d(w, v) \leq r$ for all $v$ in $V$. 

$D_1:$

$D_2:$
Landau (1909-1966)
- Dominance Relations in Animal Societies
Tournament

a digraph in which every two vertices are joined by \textit{exactly one} arc.
$k(r, D)$

$= \# \text{ of } r\text{-kings in } D$

Landau (1953)

In any tournament $T$, any vertex with \textit{highest} outdegree (\textit{score}) is a \textit{2-king}. Thus, $k(2, T) \geq 1$. 
Proof of Landau’s Observation

\[ od(v) > od(w) \]

contradiction
Questions

- If \( w \) is a 2-\textit{king}, must \( w \) have the \textit{highest score}? 
- Are there tournaments \( T \) s.t. \( k(2, T) = 1 \)?
- ♠ A \textit{source} is a vertex with zero indegree.

Every \( T \) with a \textit{source} contains a \textit{unique} 2-\textit{king}. 
• Is the *converse* of the above true?
• Under what conditions for $T$ that $k(2, T) \geq 2$?
• Are there tournaments $T$ s.t. $k(2, T) = 2$?
• What is the *best lower bound* for $k(2, T)$ if $T$ contains no source?
Moon’s Observation (1962)

Let $T$ be a tournament with no sources. Then every 2-king is dominated by a 2-king in $T$. In particular, $k(2, T) \geq 3$. (No tournament can have exactly two 2-kings.)
Proof of Moon’s Observation

\[ \text{Proof of Moon’s Observation} \]

\[ \begin{array}{c}
\text{Proof of Moon’s Observation}
\end{array} \]
Tournament $T$ \[ k(2, T) \geq 1 \]

Digraph $D$

deleting arcs from $T$

\[ k(r, D) \geq 1 \quad r = ? \]
Multipartite Tournaments

3-partite tournament: \( T(3,2,2) \)

\[\text{No sources}\]
Let $D$ be a *multipartite tournament* with at most one source. Then

$$k(4, D) \geq 1.$$
Let $D$ be an $n$-partite tournament with no source. Then

$$k(4, D) \geq \begin{cases} 4 & n = 2 \\ 3 & n \geq 3 \end{cases}$$
Thus, 4-kings are of particular interest in multipartite tournaments. In a number of papers, several authors investigate the minimum number of 4-kings in multipartite tournaments without sources. In our view, the above theorem is the most interesting result in this direction. [p.76]
Landau: $T$ tournament

$k(2, T) \geq 1$.

Any ‘team’ version of Landau’s result for general $D$?
$S \subseteq V$ is an **independent** set if no two vertices in $S$ are joined by an arc in $D$.

$K$ is a **$r$-dominating set** of $D$

(i) $K$ is **independent** &
(ii) every vertex in $V \setminus K$ can be reached from *a vertex* in $K$ *within* $r$ *steps*.

2-dominating set
**Dicycles:**

1-dominating set

No 1-dominating set

2-dominating set

Does *any* $D$ always contain a 2-*ds*?
The **Chvátal-Lovász Theorem** (1974)

Every digraph contains a **2-dominating set**.

**Chvátal** (1946 – )
Canada Research Chair in
Combinatorial Optimization
László Lovász (09/03/1948 – )
IMU President (2007–2010)
IMO-Gold(1964, 65, 66)
Wolf Prize(1999)
Gödel Prize(2001)
Kyoto Prize(2010)

虎父无犬子

Miklos Lovasz
IMO-Silver(2007)
& Gold(2008)
The **Jacob-Meyniel Theorem** (1996)

Every digraph which contains *no* 1-dominating set contains *at least three* 2-dominating sets.
**Landau:** \( k(2, T) \geq 1 \)

**Moon:** No source, \( k(2, T) \geq 3 \)

**Petrovic-Thomassen:** \( k(4, M) \geq 1 \)

**Koh-Tan:** No source,
\[ k(4, M) \geq \{3, 4\} \]

**Chvátal-Lovász:** \( \#(2-ds, D) \geq 1 \)

**Jacob-Meyniel:** No 1-ds,
\[ \#(2-ds, D) \geq 3 \]
Research Process

Observations → Problems

proposing

Problems → Results

asking

Results → Problems

solving
Art of Problem Posing

Cantor founded set theory and introduced the concept of infinite numbers with his discovery of cardinal numbers.

“In mathematics, the art of proposing a question must be held of higher value than solving it.”

George Ferdinand Ludwig Philipp Cantor (1845 – 1918)
 Cultivate the Habit of Problem-Posing