Graph Theory

1. A graph $G$ consists of a set of vertices $V$ and a set of edges $E$. An edge in $E$ joins a pair of distinct vertices in $V$. If there is at most one edge joining a pair of vertices, then $G$ is said to be simple. Otherwise, it is a multigraph. (There are cases where an edge joins a vertex to itself. Such an edge is called a loop.)

   (a) An edge $e$ is said to be incident to a vertex $v$ if $e$ joins $v$ to another vertex.

   (b) Two vertices are adjacent if they are joined by an edge.

   (c) Two edges are adjacent if they are incident to a common vertex.

   (d) For any vertex $a$, we denote by $N(a)$ the set of vertices adjacent to $a$. This set is called the neighbours of $a$.

2. The complete graph on $n$ vertices $K_n$, is the simple graph on $n$ vertices such that each of distinct vertices is joined by an edge.

3. A subgraph $G' = (V', E')$ of a graph $G = (V, E)$ is a graph where $V' \subseteq V$ and $E' \subseteq E$.

4. For any vertex $v \in V$, the degree of $v$, denoted by $\deg v$, is the number of edges incident to $v$. If the graph is simple, then $\deg v = |N(v)|$.

5. Theorem: For any graph $G = (V, E)$,

   \[ \sum_{v \in V} \deg v = 2|E|. \]

6. Theorem: For any graph $G$, there is an even number of vertices with odd degrees.

7. A trail in a graph is a sequence

   \[ v_1e_1v_2\ldots e_{n-1}v_n \]

   where the $e_i$’s are distinct edges and each edge is adjacent with its two neighbouring vertices. The first vertex is called the initial vertex and the last vertex is called the end vertex of the trail.

8. A trail is eulerian if the initial vertex is also the end vertex and all the edges of the graph are included in the trail. A graph is eulerian if it contains an eulerian trail.

9. Theorem: Let $H$ be the subgraph formed by a trail. If the initial vertex is different from the end vertex, then their degrees are odd while the degrees of all other vertices are even. If the initial vertex coincides with the end vertex, then the degrees of every vertex in $H$ is even.

10. Theorem: Suppose $G$ is a graph in which every vertex is of even degree. Then $G$ can be decomposed into edge-disjoint cycles.

11. Theorem: A connected graph is eulerian iff every vertex is of even degree.
12. A path in a graph is a sequence

\[v_1e_1v_2 \ldots e_{n-1}v_n\]

where the \(v_i\)'s are distinct vertices and the \(e_i\)'s are distinct edges and each edge is adjacent with its two neighbouring vertices. If a path has \(n\) vertices, it is usually denoted by \(P_n\).

13. A cycle in a graph is a sequence

\[v_1e_1v_2 \ldots e_{n-1}v_n e_n v_1\]

where the \(v_i\)'s are distinct vertices and the \(e_i\)'s are distinct edges and each edge is adjacent with its two neighbouring vertices. If a cycle has \(n\) vertices, it is also known as an \(n\)-cycle and is denoted by \(C_n\).

14. A cycle is hamiltonian if it contains all the vertices of the graph. A graph is said to be hamiltonian if it contains a hamiltonian cycle.

15. A graph is connected if for every two distinct vertices \(u\) and \(v\) there is a path joining \(u\) to \(v\).

16. A tree is a connected graph with no cycles.

17. Theorem: A tree with at least two vertices has at least two vertices of degree 1.

18. Theorem: A tree with \(p\) vertices has \(p - 1\) edges.

19. Theorem: A graph with \(p\) vertices and more than \(p - 1\) edges has a cycle.

20. Theorem: A graph \(G\) is connected if and only if it contains a spanning tree. (Note: A spanning tree is a subgraph of \(G\) which is a tree and contains all the vertices of \(G\).)

21. Theorem: A connected graph with \(p\) vertices has at least \(p - 1\) edges.

22. Definition. A graph \(G = (V, E)\) is bipartite iff \(V\) can be partitioned into two sets \(A, B\) such that every edge joins a vertex in \(A\) to a vertex in \(B\). \(A\) and \(B\) are known as the partite sets. (In a bipartite graph, there are no edges in each of the partite sets.) A complete bipartite graph \(K_{p,q}\) is a bipartite graph with the partite sets \(A\) and \(B\) satisfying \(|A| = p, |B| = q\) and every vertex in \(A\) is adjacent to a vertex in \(B\).

23. Theorem: A graph is bipartite iff it contains no odd cycles.

Proof: Since a cycle that starts with a vertex in \(A\) must alternate between vertices in \(A\) and \(B\) and ends in \(A\). Thus every cycle is even. Conversely, we suppose that every cycle is even and show that the graph is bipartite. Without any loss of generality, we may assume that the graph is connected. Take any vertex \(a\). For any vertex \(x\), define \(d(x)\) to be the length of the shortest path from \(a\) to \(x\). Put \(x\) in \(A\) if \(d(x)\) is even and in \(B\) is \(d(x)\) is odd. Suppose two vertices \(d, e\), in \(A\) are adjacent. Let \(a = a_1, a_2, \ldots, a_k = d\) and \(a = b_1, \ldots, b_l = e\) be the sequence of vertices in shortest paths from \(a\) to \(b\) and to \(c\), respectively. Then \(k\) and \(l\) are of the same parity. If there exist \(i \geq 2\) and \(j \geq 2\) such that \(a_i = b_j\), then \(i = j\). Then we can consider the sequence \(b_j = a_i, a_{i+1}, \ldots, a_k = d\) and
Thus we can assume that the $a_i$'s and $b_j$'s are pairwise distinct. Then we have an odd cycle $a = a_1, \ldots, a_k, b_l, b_{l-1}, \ldots, b_1 = a$. Thus the vertices in $A$ are pairwise disjoint. Likewise, so are the vertices in $B$. Thus the graph is bipartite with $A$, $B$ the partite sets.

**Problems**

1. There are $n$ participants in a meeting. Among any group of 4 participants, there is one who knows the other three members of the group. Prove that there is one participant who knows all other participants.

2. In a group of $2n$ people, $n \geq 2$, each one knows at least $n$ other people. Prove that in this group, there are four people who can be seated at a round table so that so that each person knows both his neighbours.

3. There are $n$ people in a gathering. Some of them are mutual friends. Prove that it is possible to divide them into two groups so that each person has at least half of his friends in a different group.

4. In a group of 100 people, each one knows at least 67 other people. Prove that there exist 4 people who are mutual friends.

5. Suppose you want to modify the above problem by changing the number 100 to 1000. How should 67 be altered so that the conclusion remains the same?

6. There are 500 participants in a conference. Every participant has 400 friends. Is it possible to find a group of 6 mutual friends?

7. What is your answer if everyone has 401 friends?

8. Prove that in a group of 18 people, there is either a group of 4 mutual friend or a group of 4 mutual strangers.

9. There are 18 contestants in a tournament. In each round the contests are paired and play each other once. Prove that after 8 rounds, there are three contestant who have not played against each other.

10. Let $G$ be a graph with 10 vertices. Among any three vertices of $G$, at least two are adjacent. Find the least number of edges that $G$ can have. Find a graph with this property.

11. In an $n \times n$ matrix, the rows are pairwise distinct. Prove that there is a column, whose removal results in an $n \times (n - 1)$ matrix with the same property.

12. In a group of 1997 people, among 4 of them there is at least one who knows the other three. What is minimum of people in the group who knows everybody else?

13. At the end of a birthday party, the hostess wants to give away candies. She has 6 types of cookies. Each child is given a gift packet which contains two types of cookies.
Each type of cookie is used in combination with at least three others. Prove there are three children, who between them, have all the six types of cookies.

14. In a group of people, any two mutual friends have no common friends while any pair of mutual strangers have exactly two common friends. Prove that there are two persons in this group who have the same number of friends.

15. In a party there are $12k$ guests. Every guest knows exactly $3k + 6$ other guests. Suppose that if $x$ knows $y$, then $y$ knows $x$ too. For every two guests $x$ and $y$ in this party there are exactly $n$ guests who know both $x$ and $y$. ($n$ is a constant). Prove that

$$9k^2 + (33 - 12n)k + (30 + n) = 0$$

and then solve for $n$ and $k$.

16. Given $n$ points on the plane such that the distance between every pair of points is at least 1, prove that there are at most $3n$ pairs of points such that the distance between two points in each pair is 1.

17. There are 9 mathematicians in a meeting. It was discovered that each of them can speak at most three languages and among any three of them at least two can speak a common language. Prove that three are three mathematicians who speak a common language.

18. Can you place one number chosen from \{0, 1, \ldots, 9\} on each of the vertices of polygon with 45 sides, so that for every pair of of integers, $a$, $b$, $0 \leq a < b \leq 9$, there is a side of the polygon whose ends have the numbers $a$ and $b$?

19. What is your answer if in the previous problem, the numbers 45 and 9 are replace by 55 and 10?

20. There are 3 schools each with $n$ students. Every students knows $n + 1$ students from the other two schools. Prove that it is possible to find one student from each of the schools such that the three students know each other.

21. There are $n$ people in a room. Any group of $m(\geq 3)$ people in the room have a unique common friend. Can you determine $n$ in terms of $m$?

22. (IMO 1990) Let $n \geq 3$ and consider a set $E$ of $2n - 1$ distinct points on a circle. Suppose that exactly $k$ of these points are to be colored black. Such a coloring is good if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly $n$ points from $E$. Find the smallest value of $k$ so that every such coloring of $k$ points of $E$ is good.

23. Find the smallest positive integer $n$ such that in any set of $n$ irrational numbers, there are three numbers such that the sum of every two of them is again irrational.
In this section we discuss colourings of the edges of $K_n$ using two colours. Generally we want to know whether there is an $m < n$ so that there is a monochromatic $K_m$ in the original $K_n$. (Monochromatic here means that all its edges have the same colour.)

1. Theorem. In any colouring of the edges of $K_6$ using two colours, there is always a monochromatic $K_3$.

2. Theorem. There is a colouring of the edges of $K_5$ using two colours such that there is no monochromatic $K_3$.

3. Ramsey numbers: One can generalize the above to colourings using three or more colours. In general, one can define $r_{k}$ to be the smallest $n$ such that in every colouring of the edges of $K_n$ using $k$ colours, there is always a monochromatic $K_3$. $r_k$ is known as Ramsey numbers.

4. From 1 and 2, we have $r_2 = 6$.

5. Theorem. For every positive integer $k \geq 2$, we have

$$r_k \leq k(r_{k-1} - 1) + 2.$$

Proof. We shall prove the theorem by induction on $k$. We know that $r_1 = 3$ and $r_2 = 6$. Thus the result holds for $k = 2$. Now we assume that the results holds for $k$. We’ll prove that it also holds for $k + 1$.

Write $n = (k + 1)(r_k - 1) + 2$. Take any colouring of the edges of $K_n$ using $k + 1$ colours. For an arbitrary vertex $x$ of $K_n$, among the edges incident to $x$, $r_k$ of them have the same colour, say colour 1. Among the neighbours of $x$, if there are two vertices joined by an edge coloured using colour 1, then there is a monochromatic triangle with colour 1. If not, then the edges joining the $r_k$ neighbours of $x$ are coloured using colours 2, \ldots, $k$. By the induction hypothesis, there is a monochromatic triangle.

6. It is known at $r_3 = 17$. But $r_k$ for $k \geq 4$ are still unknown.

7. If we restrict the number of colours used to two, say red and blue, then there is another form of Ramsey’s theorem. Given $p$ and $q$, there exists $n$ such that if one colours the edges of $K_n$ red or blue, there is either a red $K_p$ or a blue $K_q$. The minimum value of such $n$ is denoted by $r(p, q)$. Thus $r(3, 3) = r_2$, $r(3, 3, 3, 3) = r_4$, etc.

8. Theorem:

$$r(p, q) \leq r(p, q - 1) + r(p - 1, q)$$

$$r(p, q) \leq r(p, q - 1) + r(p - 1, q) - 1$$

if both $r(p, q - 1) + r(p - 1, q)$ are even

Proof: Let $n = r(p, q - 1) + r(p - 1, q)$. We shall any colouring of the edges of $K_n$ using red and blue contains either a red $K_p$ or a blue $K_q$). Let $v$ be any vertex of $K_n$ and
A be the set of vertices joined to v by red edges and B the set of vertices joint to v by blue edges. We have either |A| ≥ r(p − 1, q) or |B| ≥ r(p, q − 1). In the former, A contains a red $K_{p-1}$ or a blue $K_q$. Thus $K_n$ contains a red $K_p$ or a blue $K_q$. In the latter B contains a red $K_p$ or a blue $K_{q-1}$. The same conclusion follows.

If both $r(p, q − 1) + r(p − 1, q)$ are even, let $m = r(p, q − 1) + r(p − 1, q) − 1$. By considering the red edges only, we have $\sum N_r(v)$ is twice the number of red edges, where $N_r(v)$ is the number of red edges joined to v. Thus there is a vertex w with $N_r(w)$ even. Let A be the set of vertices joined to w by red edges and B the set of vertices joined to w by blue edges. Then |A| ≥ $r(p − 1, q) − 1$ or |B| ≥ $r(p, q − 1)$. But |A| is even, so we have |A| ≥ $r(p − 1, q)$ or |B| ≥ $r(p, q − 1)$. Thus the same conclusion follows as in the previous case.

**Problems**

1. Given 6 lines in space such that no three lie on the same plane, prove that there exist three lines which satisfy one of the following:
   (i) they are pairwise skew;
   (ii) they are parallel;
   (iii) they are concurrent.

2. Given 6 points in the plane, no three collinear, prove that there are two triangles whose vertices are among the 6 given points such that the longest of one of them is the shortest side of the other.

3. Prove that any colouring of the edges of $K_6$ using two colours produces two monochromatic triangles.

4. Prove that among a group of 9 people there are 3 mutual friends or there are 4 mutual strangers. (This means $r(3, 4) ≤ 9$.)

5. Prove that $r(3, 4) = 9$.

6. Prove that among a group of 14 people there are 3 mutual friends or there are 5 mutual strangers.

7. Colour the integral points $(x, y)$, where $1 ≤ x ≤ 16$ and $1 ≤ y ≤ 9$, using three colours. Prove that there exists a monochromatic rectangle whose sides are parallel to the axes.

8. Colour the points on the plane either red or blue. Prove that there exists a monochromatic equilateral triangle whose sides are of length either 1 or $\sqrt{3}$.

9. Colour the points on the plane using three colours. Prove that there are two points with the same colour and are unit distance apart.