Problem 1. Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by $s_A$. Let $n_A$ denote the number of pairs $(i, j)$ with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides $s_A$. Find all sets $A$ of four distinct positive integers which achieve the largest possible value of $n_A$.

Problem 2. Let $S$ be a finite set of at least two points in the plane. Assume that no three points of $S$ are collinear. A windmill is a process that starts with a line $\ell$ going through a single point $P \in S$. The line rotates clockwise about the pivot $P$ until the first time that the line meets some other point belonging to $S$. This point, $Q$, takes over as the new pivot, and the line now rotates clockwise about $Q$, until it next meets a point of $S$. This process continues indefinitely.

Show that we can choose a point $P$ in $S$ and a line $\ell$ going through $P$ such that the resulting windmill uses each point of $S$ as a pivot infinitely many times.

Problem 3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a real-valued function on the set of real numbers that satisfies

$$f(x + y) \leq yf(x) + f(f(x))$$

for all real numbers $x$ and $y$. Prove that $f(x) = 0$ for all $x \leq 0$. 

Language: English

Time: 4 hours 30 minutes.

Each problem is worth 7 marks.