The 37th Chinese Mathematical Olympiad

Fuzhou, Fujian

Day I

December 21, 2021; 8:00–12:30

1. Let a and b be two positive real numbers, and AB a segment of length a on a plane. Let C and D be two moving points on this plane so that $ABCD$ is a nondegenerate convex quadrilateral with $BC = CD = b$ and $DA = a$. It is easy to see that there exists a circle, with center I, tangent to all four sides of the quadrilateral $ABCD$.

Find the precise locus of the point I.

2. Find the largest real number λ with the following property: for any positive real numbers p, q, r, s, there exists a complex number $z = a + b\mathrm{i}$ $(a, b \in \mathbb{R})$ such that

$$|b| \geqslant \lambda\,|a| \quad \text{and} \quad (pz^3 + 2qz^2 + 2rz + s) \cdot (qz^3 + 2pz^2 + 2sz + r) = 0.$$

3. Find all the integers a such that there exists a set X of 6 integers satisfying the following conditions: for every $k = 1, 2, \ldots, 36$, there exist $x, y \in X$ such that $ax + y - k$ is divisible by 37.

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Day II
December 22, 2021; 8:00–12:30

4. A conference is attended by n ($n \geqslant 3$) scientists. Each scientist has some friends in this conference (being friends is mutual, and no one is considered a friend of him/herself). Assume that, no matter how we separate these scientists into two nonempty groups, there always exist two scientists from the same group who are friends, and there always exist two scientists from different groups who are friends.

A proposal is introduced on the first day of the conference. Each of the scientists' opinion on the proposal can be represented by a nonnegative integer. Everyday from the second day onward, each scientist's opinion on the proposal is changed to the integer part of the average of all his/her friends' opinions on the proposal from the previous day.

Prove that, after a period of time, all these scientists have the same opinion on this proposal.

5. We know that in *straightedge and compass constructions*, there are only two types of 1-dimensional geometric objects: circles and straight lines. Start with a blank piece of paper, on which only two points with distance 1 are given. Prove that one can use straightedge and compass to construct on this paper a straight line, and two points on it of distance $\sqrt{2021}$ such that, in the process of constructing them, the total number of circles and straight lines appeared is less than or equal to 10.

Remark: Explicit steps of the construction should be given. Please label the circles and straight lines in the order they appear. If your construction involves more than 10 circles and lines, a partial credit may be awarded depending on the total number of circles and lines.)

6. For integers $0 \leqslant a \leqslant n$, let $f(n,a)$ denote the number of coefficients in the expansion of $(x+1)^a(x+2)^{n-a}$ that are divisible by 3. For example, $(x+1)^3(x+2)^1 = x^4+5x^3+9x^2+7x+2$, so $f(4,3)=1$. For each positive integer n, let $F(n)$ denote the minimum of $f(n,0), f(n,1), \ldots, f(n,n)$.

(1) Prove that there exist infinitely many positive integers n such that $F(n) \geqslant \frac{n-1}{3}$.

(2) Prove that for any positive integer n, we have $F(n) \leqslant \frac{n-1}{3}$.