The 40th China Mathematical Olympiad

Day 1

1. Let $\alpha > 1$ be an irrational number and L be a integer such that $L > \frac{\alpha^2}{\alpha-1}$. A sequence $x_1, x_2, \cdots$ satisfies that $x_1 > L$ and for all positive integers n,

$$x_{n+1} = \begin{cases} \lfloor \alpha x_n \rfloor & \text{if } x_n \leqslant L \\ \lfloor \frac{x_n}{\alpha} \rfloor & \text{if } x_n > L \end{cases}.$$

Prove that

(i) $\{x_n\}$ is eventually periodic.

(ii) The eventual fundamental period of $\{x_n\}$ is an odd integer which doesn't depend on the choice of x_1.

2. Let ABC be a triangle with incenter I. Denote the midpoints of AI, AC and CI by L, M and N respectively. Point D lies on segment AM such that $BC = BD$. Let the incircle of triangle ABD be tangent to AD and BD at E and F respectively. Denote the circumcenter of triangle AIC by J, and the circumcircle of triangle JMD by ω. Lines MN and JL meet ω again at P and Q respectively. Prove that PQ, LN and EF are concurrent.

3. Let $a_1, a_2, \ldots, a_n$ be integers such that $a_1 > a_2 > \cdots > a_n > 1$. Let $M = \operatorname{lcm}(a_1, a_2, \ldots, a_n)$. For any finite nonempty set X of positive integers, define

$$f(X) = \min_{1 \leqslant i \leqslant n} \sum_{x \in X} \left\{ \frac{x}{a_i} \right\}.$$

Such a set X is called minimal if for every proper subset Y of it, $f(Y) < f(X)$ always holds.

Suppose X is minimal and $f(X) \geqslant \frac{2}{a_n}$. Prove that

$$|X| \leqslant f(X) \cdot M.$$

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Day 2

4. The fractional distance between two points (x_1, y_1) and (x_2, y_2) is defined as

$$\sqrt{\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2},$$

where $\|x\|$ denotes the distance between x and its nearest integer. Find the largest real r such that there exists four points on the plane whose pairwise fractional distance are all at least r.

5. Let p be a prime number and f be a bijection from $\{0, 1, \ldots, p-1\}$ to itself. Suppose that for integers $a, b \in \{0, 1, \ldots, p-1\}$, $|f(a) - f(b)| \leqslant 2024$ if $p \mid a^2 - b$. Prove that there exists infinite many p such that there exists such an f and there also exists infinite many p such that there doesn't exist such an f.

6. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $\sum_{i=1}^n a_i = n$, $\sum_{i=1}^n a_i^2 = 2n$, $\sum_{i=1}^n a_i^3 = 3n$.

(i) Find the largest constant C, such that for all $n \geqslant 4$,

$$\max\{a_1, a_2, \ldots, a_n\} - \min\{a_1, a_2, \ldots, a_n\} \geqslant C.$$

(ii) Prove that there exists a positive constant C_2, such that

$$\max\{a_1, a_2, \ldots, a_n\} - \min\{a_1, a_2, \ldots, a_n\} \geqslant C + C_2 n^{-\frac{3}{2}},$$

where C is the constant determined in (i).