The 38^{th} Chinese Mathematical Olympiad

Shenzhen, Guangdong

Day One

8:00–12:30 Dec. 29, 2022

1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive real numbers such that, for any positive integer n,

$$a_{n+1} = a_n - \frac{1}{1+\sum\limits_{i=1}^{n} \frac{1}{a_i}}, \quad \text{and} \quad b_{n+1} = b_n + \frac{1}{1+\sum\limits_{i=1}^{n} \frac{1}{b_i}}.$$

(1) If $a_{100}b_{100} = a_{101}b_{101}$, find the value of $a_1 - b_1$.

(2) If $a_{100} = b_{99}$, which one of $a_{100} + b_{100}$ and $a_{101} + b_{101}$ is larger?

2. Fix an equilateral triangle ABC with side length 1. We call $(\triangle DEF, \triangle XYZ)$ a *good triangle pair* if the points D, E, and F lie in the interior of segments BC, CA, and AB, respectively, the points X, Y, and Z lie on the lines BC, CA, and AB, respectively, and they satisfy the conditions

$$\frac{DE}{20} = \frac{EF}{22} = \frac{FD}{38}, \quad \text{and} \quad DE \perp XY,\ EF \perp YZ,\ FD \perp ZX.$$

As $(\triangle DEF, \triangle XYZ)$ runs through all good triangle pairs, determine all possible values of $\frac{1}{S_{\triangle DEF}} + \frac{1}{S_{\triangle XYZ}}$.

3. Fix two positive integers m and n. Fix a way to color the vertices of a regular $(2m+2n)$-gon so that $2m$ of them are black and the other $2n$ are white. Define the *coloring distance* $d(B,C)$ between two black points B and C to be the lesser of the numbers of white points on either side of the line BC; similarly, define the *coloring distance* $d(W,X)$ between two white points W and X to be the lesser of the numbers of black points on either side of the line WX.

A *black pairing scheme* $\mathcal{B}$ means to label the $2m$ black points as $B_1, \ldots, B_m, C_1, \ldots, C_m$, such that no two segments from $B_1C_1, \ldots, B_mC_m$ intersect each other. For each such $\mathcal{B}$, put

$$P(\mathcal{B}) = \sum_{i=1}^{m} d(B_i, C_i).$$

A *white pairing scheme* $\mathcal{W}$ means to label the $2n$ white points as $W_1, \ldots, W_n, X_1, \ldots, X_n$, such that no two segments from $W_1X_1, \ldots, W_nX_n$ intersect each other. For each such $\mathcal{W}$, put

$$P(\mathcal{W}) = \sum_{i=1}^{n} d(W_i, X_i).$$

Prove that, regardless of how the $2m+2n$ vertices are colored, we always have

$$\max_{\mathcal{B}} P(\mathcal{B}) = \max_{\mathcal{W}} P(\mathcal{W}),$$

where the maxima are taken over all possible black pairing schemes and all possible white pairing schemes, respectively.

2022 年全国中学生数学奥林匹克竞赛 (决赛)

广东　深圳

第一天

2022 年 12 月 29 日 8:00–12:30

1. 设正实数序列 $\{a_n\}$, $\{b_n\}$ 满足: 对任意正整数 n, 均有

$$a_{n+1}=a_n-\frac{1}{1+\sum\limits_{i=1}^{n}\frac{1}{a_i}},\quad b_{n+1}=b_n+\frac{1}{1+\sum\limits_{i=1}^{n}\frac{1}{b_i}}.$$

(1) 若 $a_{100}b_{100}=a_{101}b_{101}$, 求 a_1-b_1 的值.

(2) 若 $a_{100}=b_{99}$, 比较 $a_{100}+b_{100}$ 与 $a_{101}+b_{101}$ 的大小.

2. 给定一个边长为 1 的正三角形 ABC. 称 $(\triangle DEF, \triangle XYZ)$ 是一个好三角形对, 如果点 D, E, F 分别在线段 BC, CA, AB 内部, 点 X, Y, Z 分别在直线 BC, CA, AB 上, 满足

$$\frac{DE}{20}=\frac{EF}{22}=\frac{FD}{38},\ \text{且}\ DE\perp XY,\ EF\perp YZ,\ FD\perp ZX.$$

当 $(\triangle DEF, \triangle XYZ)$ 取遍所有好三角形对时, 求 $\frac{1}{S_{\triangle DEF}}+\frac{1}{S_{\triangle XYZ}}$ 的所有可能值.

3. 给定正整数 m 和 n. 将正 $2m+2n$ 边形的 $2m$ 个顶点染黑色, 其余 $2n$ 个顶点染白色. 定义两个黑点 B, C 的染色距离 $d(B,C)$ 为直线 BC 两侧的白点数目的较小者; 定义两个白点 W, X 的染色距离 $d(W,X)$ 为直线 WX 两侧的黑点数目的较小者.

一个黑点配对方案 $\mathscr{B}$ 是指将所有 $2m$ 个黑点标记为 $B_1,\cdots,B_m,C_1,\cdots,C_m$, 使得 m 条线段 B_iC_i $(1\leqslant i\leqslant m)$ 两两不相交. 对任一黑点配对方案 $\mathscr{B}$, 记

$$P(\mathscr{B})=\sum_{i=1}^{m}d(B_i,C_i).$$

一个白点配对方案 $\mathscr{W}$ 是指将所有 $2n$ 个白点标记为 $W_1,\cdots,W_n,X_1,\cdots,X_n$, 使得 n 条线段 W_iX_i $(1\leqslant i\leqslant n)$ 两两不相交. 对任一白点配对方案 $\mathscr{W}$, 记

$$P(\mathscr{W})=\sum_{i=1}^{n}d(W_i,X_i).$$

证明: 无论顶点的染色方式如何, 均有

$$\max_{\mathscr{B}}P(\mathscr{B})=\max_{\mathscr{W}}P(\mathscr{W}),$$

其中等式两边的最大值分别在所有可能的黑点配对方案 $\mathscr{B}$ 和白点配对方案 $\mathscr{W}$ 中选取.

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Day Two

8:00–12:30 Dec. 30, 2022

4. Find the minimal integer $n \geq 3$ such that there exist n points $A_1, A_2, \ldots, A_n$ on a plane with no three being colinear, such that for every $1 \leq i \leq n$, the midpoint of the segment A_iA_{i+1} is contained in the segment A_jA_{j+1} for some $j \neq i$. Here, $A_{n+1} = A_1$.

5. Prove that there exists a positive number C such that the following statement holds: for any infinite arithmetic progression $a_1, a_2, a_3, \ldots$ of positive integers, if the greatest common divisor of a_1 and a_2 is square-free, then there exists some positive integer $m \leq C \cdot a_2^2$ such that a_m is square-free.

Remark: We call a positive integer N square-free, *if it is not divisible by any square that is strictly larger than* 1.

6. There are some direct one-way flights among $n \geq 8$ airports. Between any two airports a and b, there is at most one direct one-way flight from a to b (it is possible to have direct one-way flights both from a to b and from b to a). Suppose that, for any set A consisting of some airports with $1 \leq |A| \leq n-1$, there are at least $4 \cdot \min\{|A|, n-|A|\}$ flights in total departing from the airports in A and arriving at the airports not in A.

Prove that for any airport x, one can depart from x and take no more than $\sqrt{2n}$ flights to return to x.

2022 年全国中学生数学奥林匹克竞赛 (决赛)

广东　深圳

第二天

2022 年 12 月 30 日 8:00–12:30

4. 求最小的整数 $n \geqslant 3$, 满足: 平面上存在 n 个点 $A_1, A_2, \cdots, A_n$, 其中任意三点不共线, 且对任意 $1 \leqslant i \leqslant n$, 存在 $1 \leqslant j \leqslant n$ $(j \neq i)$, 使得线段 A_jA_{j+1} 经过线段 A_iA_{i+1} 的中点. 这里 $A_{n+1} = A_1$.

5. 证明存在正数 C, 使得如下结论成立:

对任意一个无穷多项的正整数等差数列 $a_1, a_2, a_3, \cdots$, 若 a_1 和 a_2 的最大公约数无平方因子, 则存在正整数 $m \leqslant C \cdot a_2^2$, 使得 a_m 无平方因子.

注: 称正整数 N 无平方因子, 若它不被任何大于 1 的平方数整除.

6. 有 $n (n \geqslant 8)$ 座机场, 某些机场之间有单向直达航线. 对任意两座机场 a, b, 从 a 飞往 b 的单向直达航线至多一条 (可能同时有从 a 飞往 b 的和从 b 飞往 a 的单向直达航线). 已知对任意由若干座机场构成的集合 A $(1 \leqslant |A| \leqslant n-1)$, 都有至少 $4 \cdot \min\{|A|, n-|A|\}$ 条单向直达航线从 A 中的机场飞往 A 之外的机场.

证明: 对任意一座机场 x, 都可以从 x 出发, 经过不超过 $\sqrt{2n}$ 条单向直达航线回到机场 x.