IMO 2017
RIO DE JANEIRO - BRAZIL
58th International Mathematical Olympiad

English (eng), day 1

Tuesday, July 18, 2017

Problem 1. For each integer $a_0 > 1$, define the sequence $a_0, a_1, a_2, \ldots$ by:

$$a_{n+1} = \begin{cases} \sqrt{a_n} & \text{if } \sqrt{a_n} \text{ is an integer,} \\ a_n + 3 & \text{otherwise,} \end{cases} \qquad \text{for each } n \geqslant 0.$$

Determine all values of a_0 for which there is a number A such that $a_n = A$ for infinitely many values of n.

Problem 2. Let $\mathbb{R}$ be the set of real numbers. Determine all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x and y,

$$f\left(f(x)f(y)\right) + f(x+y) = f(xy).$$

Problem 3. A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0, are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order.

(i) The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.

(ii) A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.

(iii) The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.

Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that after 10^9 rounds she can ensure that the distance between her and the rabbit is at most 100?

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Wednesday, July 19, 2017

Problem 4. Let R and S be different points on a circle Ω such that RS is not a diameter. Let ℓ be the tangent line to Ω at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen on the shorter arc RS of Ω so that the circumcircle Γ of triangle JST intersects ℓ at two distinct points. Let A be the common point of Γ and ℓ that is closer to R. Line AJ meets Ω again at K. Prove that the line KT is tangent to Γ.

Problem 5. An integer $N \geqslant 2$ is given. A collection of $N(N+1)$ soccer players, no two of whom are of the same height, stand in a row. Sir Alex wants to remove $N(N-1)$ players from this row leaving a new row of $2N$ players in which the following N conditions hold:

(1) no one stands between the two tallest players,

(2) no one stands between the third and fourth tallest players,

$\vdots$

(N) no one stands between the two shortest players.

Show that this is always possible.

Problem 6. An ordered pair (x, y) of integers is a *primitive point* if the greatest common divisor of x and y is 1. Given a finite set S of primitive points, prove that there exist a positive integer n and integers $a_0, a_1, \ldots, a_n$ such that, for each (x, y) in S, we have:

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n = 1.$$