

Tuesday, July 16, 2019

Problem 1. Let \mathbb{Z} be the set of integers. Determine all functions $f: \mathbb{Z} \rightarrow \mathbb{Z}$ such that, for all integers a and b ,

$$f(2a) + 2f(b) = f(f(a + b)).$$

Problem 2. In triangle ABC , point A_1 lies on side BC and point B_1 lies on side AC . Let P and Q be points on segments AA_1 and BB_1 , respectively, such that PQ is parallel to AB . Let P_1 be a point on line PB_1 , such that B_1 lies strictly between P and P_1 , and $\angle PP_1C = \angle BAC$. Similarly, let Q_1 be a point on line QA_1 , such that A_1 lies strictly between Q and Q_1 , and $\angle CQ_1Q = \angle CBA$.

Prove that points P , Q , P_1 , and Q_1 are concyclic.

Problem 3. A social network has 2019 users, some pairs of whom are friends. Whenever user A is friends with user B , user B is also friends with user A . Events of the following kind may happen repeatedly, one at a time:

Three users A , B , and C such that A is friends with both B and C , but B and C are not friends, change their friendship statuses such that B and C are now friends, but A is no longer friends with B , and no longer friends with C . All other friendship statuses are unchanged.

Initially, 1010 users have 1009 friends each, and 1009 users have 1010 friends each. Prove that there exists a sequence of such events after which each user is friends with at most one other user.

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Problem 4. Find all pairs (k, n) of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}).$$

Problem 5. The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H , then he turns over the k^{th} coin from the left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \rightarrow HHT \rightarrow HTT \rightarrow TTT$, which stops after three operations.

- (a) Show that, for each initial configuration, Harry stops after a finite number of operations.
- (b) For each initial configuration C , let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C .

Problem 6. Let I be the incentre of acute triangle ABC with $AB \neq AC$. The incircle ω of ABC is tangent to sides BC , CA , and AB at D , E , and F , respectively. The line through D perpendicular to EF meets ω again at R . Line AR meets ω again at P . The circumcircles of triangles PCE and PBF meet again at Q .

Prove that lines DI and PQ meet on the line through A perpendicular to AI .