44th International Mathematical Olympiad

Tokyo, Japan, July 2003

1. Let A be a 101-element subset of the set $S = \{1, 2, 3, \ldots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\} \quad \text{for each } j = 1, 2, \ldots, 100$$

are pairwise disjoint.

Soln. Let $D = \{x - y \mid x, y \in A, x > y\}$. If $A_i \cap A_j \neq \emptyset$, then there exist $x, y \in A$, such that $x + t_i = y + t_j$ or $0 \neq t_i - t_j = y - x \in D$ (without loss of generality, assume that $y > x$.

Take $t_1 = 1$ and choose the smallest $t_2 \in S$ such that $t_2 \notin D_1 = \{x + t_1 \mid x \in D\}$. In general, if $t_1, \ldots, t_i$, $i < 100$ has been chosen, choose the smallest $t_{i+1} \in S$ such that $t_{i+1} \notin D_k = \{x + t_k \mid x \in D\}$, $k = 1, \ldots, i$. Since $\sum |D_k| \leq i\binom{101}{2} \leq 5050i < 100000$, such a choice can be made. It's clear that $t_i - t_j \notin D$, $100 \geq i > j \geq 1$. So $A_i \cap A_j = \emptyset$.

2. Find all pairs of positive integers (a, b) such that the number

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is also a positive integer.

Soln. Let

$$\frac{a^2}{2ab^2 - b^3 + 1} = k.$$

Then

$$a^2 - 2kb^2a + k(b^3 - 1) = 0$$

This discriminant D is the square of some integer d, i.e.,

$$D = (2b^2k - b)^2 + 4k - b^2 = d^2.$$

If $4k - b^2 = 0$, we get

$$a = 2b^2k - \frac{b}{2} \quad \text{or} \quad \frac{b}{2}$$

If $4k - b^2 > 0$, we get

$$d^2 - (2b^2k - b)^2 = 4k - b^2 \geq 2(2b^2k - 2b + 1),$$

i.e.,

$$4k(b^2 - 1) + (b^2 - 1) \leq 0.$$

This implies $b = 1$. If $4k - b^2 < 0$, then $(2b^2k - b)^2 - d^2 = b^2 - 4k$. But $(2b^2k - b)^2 - (2b^2k - b - 1)^2 = 2(2b^2k - b) - 1$. Since

$$2(2b^2k - b) - 1 - (b^2 - 4k) = b^2(4k - 3) + 2b(b - 1) + (4k - 1) > 0,$$

we get a contradiction and so there is no solution. So the solutions are:

$$(a, b) = (2k, 1), (k, 2k), (8k^4 - k, 2k)$$

3. Given is a convex hexagon with the property that the segment connecting the middle points of each pair of opposite sides in the hexagon is $\frac{\sqrt{3}}{2}$ times the sum of those sides' sum.

Prove that the hexagon has all its angles equal to $120°$.

Soln. Let $a_1, , \ldots, a_6$ be vectors for the vertices. Then we know

$$|(a_1 + a_2)/2 - (a_4 + a_5)/2| = \sqrt(3)/2(|a_1 - a_2| + |a_4 - a_5|) \geq \sqrt(3)/2(|a_1 - a_2 + a_5 - a_4|)$$

(with equality iff the two sides are parallel). So

$$|(a_1 - a_4) + (a_2 - a_5)| \geq \sqrt(3)|(a_1 - a_4) - (a_2 - a_5)|.$$

Let $r_1 = a_1 - a_4$, $r_2 = a_2 - a_5$, $r_3 = a_3 - a_6$ (i.e the diagonals) We get

$$|r_1 + r_2| \geq \sqrt(3)|r_1 - r_2| \tag{1}$$

and similarly

$$|r_2 + r_3| \geq \sqrt(3)|r_2 - r_3| \tag{2}$$
$$|r_1 - r_3| \geq \sqrt(3)|r_1 + r_3| \tag{3}$$

(note the sign changes here) with equality iff the corresponding sides are parallel. By squaring (1) we get

$$r_1 \cdot r_2 \geq \frac{1}{4}(|r_1|^2 + |r_2|^2).$$

So if x is the angle between r_1 and r_2, we see that

$$\cos x \geq \frac{|r_1|^2 + |r_2|^2}{4|r_1||r_2|} \geq \frac{1}{2}$$

(by AM-GM) with equality iff $|r_1| = |r_2|$. So $0 \leq x \leq \pi/3 = 60°$. Similarly angle between r_2 and r_3 is between 0 and $\pi/3$ whereas angle between r_1 and r_3 is $\geq 2\pi/3$. So this can only hold if all equalities hold. Therefore $|r_1| = |r_2| = |r_3|$, and the diagonals intersect pairwise at 60° and opposite sides are parallel. Now let $ABCDEF$ be the hexagon. (Then draw a line DG parallel to EB and equal to that segment, i.e., $EDGB$ is a convex parallelogram. So $ED \parallel BG$ but also $ED \parallel AB$, so A, B, G are collinear. Then $\angle ADG = 60^\circ$, since $DG \parallel EB$ and AD intersects EB at 60°. Also $AD = EB = DG$. So $\triangle ADG$ is equilateral. So $\angle DAG = \angle DAB = 60^\circ$. Similarly $\angle DAF = 60^\circ$. So add them to get $\angle FAB = 120^\circ$. Similarly for all the other angles.

4. Given is a cyclic quadrilateral $ABCD$ and let P, Q, R be feet of the altitudes from D to AB, BC and CA respectively. Prove that if $PR = RQ$ then the interior angle bisectors of $\angle ABC$ and $\angle ADC$ are concurrent on AC.

Soln. The well-known 'Pedal Triangle Trick' is "For any point D, let X, Y, Z be feet of the altitudes from D to AB, BC, CA. Then, $XZ = (DA \cdot BC)/2r$, etc, where r is the circumradius of ABC." The proof is very easy, since D, A, Y, Z lie on a circle with diameter DA, by the law of sines, $XZ = DA \sin A = DA \cdot BC/2r$.

By PTT, $PR = RQ$ implies $DA \cdot BC/2r = DC \cdot AB/2r$, so, $CD/DA = BC/AB$ implies the results.

5. Let $x_1 \le x_2 \le \cdots \le x_n$ be real numbers, $n > 2$.

(a) Prove the following inequality:

$$\Big(\sum_{i,j} |x_i - x_j|\Big)^2 \le \frac{2(n^2-1)}{3} \sum_{i,j} (x_i - x_j)^2.$$

(b) Prove that the equality in the inequality above is obtained if and only if the sequence (x_k) is an arithmetic progression.

Soln. The inequality is equivalent to

$$\Big(\sum_{i<j} |x_i - x_j|\Big)^2 \le \frac{(n^2-1)}{3} \sum_{i<j} (x_i - x_j)^2.$$

Let L be the LHS and R be the sum on the RHS. Further, let $a_i = x_{i+1} - x_i$, $i = 1, \ldots, n-1$. Then

$$L = [1(n-1)a_1 + 2(n-2)a_2 + \cdots + (n-1)1a_{n-1}]^2$$

$$R = \sum_{i=1}^{n-1} [a_i^2 + (a_i + a_{i+1})^2 + \cdots + (a_i + a_{i+1} + \cdots + a_{n-1})^2]$$

Using the fact that $1^2 + 2^2 + \cdots + k^2 = 2\binom{k+1}{3} + \binom{k+1}{2}$ and that $\binom{i}{i} + \binom{i+1}{i} + \cdots + \binom{k}{i} = \binom{k+1}{i+1}$, we get, by Cauchy-Schwarz Inequality,

$$\begin{aligned}
\frac{n^2(n^2-1)}{12} \times R &= \Big(\sum_{i=1}^{n-1} (1^2 + 2^2 + \cdots + (n-i)^2)\Big) \times R \\
&\ge \Big(\sum_{i=1}^{n-1} [1a_i + 2(a_i + a_{i+1}) + \cdots + (n-i)(a_i + a_{i+1} + \cdots + a_{n-1})\Big)^2 \\
&= \frac{n^2}{4} L
\end{aligned}$$

Thus the inequality follows. Since equality holds iff $(a_1+a_2)/a_1 = 2$, $(a_1+a_2+a_3)/a_1) = 3$, etc, we see that inequality holds iff $a_1 = a_2 = \cdots = a_{n-1}$, in other words, when the (x_k) is an AP.

6. Prove that for each given prime p there exists a prime q such that $n^p - p$ is not divisible by q for each positive integer n.

Soln. Since

$$\frac{p^p - 1}{p-1} = 1 + p + p^2 + \cdots + p^{p-1} \equiv p + 1 \pmod{p^2}$$

there is a prime divisor q of $(p^p-1)/(p-1)$ which is not congruent to 1 modulo p^2. We claim that q has the desired properties. Assume, on the contrary, that there exists n such that $n^p \equiv p \pmod{q}$. Then we have $n^{p^2} \equiv p^p \equiv 1 \pmod{q}$ by the definition of q. On the other hand, from Fermat's Little Theorem, $n^{q-1} \equiv 1 \pmod{q}$. Since $p^2 \nmid q-1$, we have $(p^2, q-1) \mid p$, which leads to $n^p \equiv 1 \pmod{q}$. Hence we have $p \equiv 1 \pmod{q}$. However, this implies $1 + p + \cdots + p^{p-1} \equiv p \pmod{q}$. From the definition of q, this leads to $p \equiv 0 \pmod{q}$, a contradiction.