45th International Mathematical Olympiad

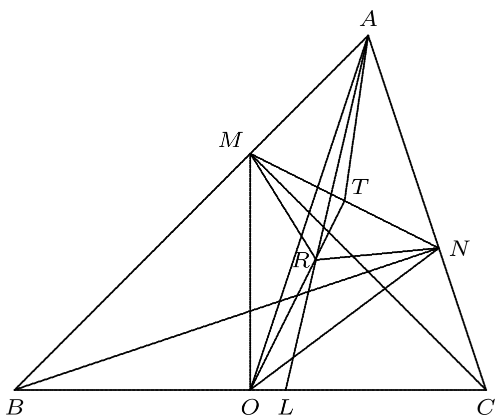
Athens, Greece, July 2004

1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of the side BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

Soln. The radical centre of the three circles is A. Thus the radical axis of circles BMR and CNR is AR. Thus we only need to show that $BMRL$ and $CNRL$ are cyclic, where $L = AR \wedge BC$. We have

$$OM = ON \quad \text{and} \quad \angle NOR = \angle MOR \quad \Rightarrow \quad MR = RN.$$

Also $\angle MAR = \angle NAR$, $AR = AR$ and $MR = RN$ imply either (i) $\triangle AMR \equiv \triangle ANR$ which in turn implies that $AM = AN$ (impossible as $AB \neq AC$); or (ii) $\angle AMR + \angle ANR = 180°$ which in turn implies that $AMRN$ is cyclic. (This fact can also be obtained by noting that the perpendicular bisector of MN and the angle bisector of $\angle MAN$ meet at R.) This then implies that $\angle ARN = \angle AMN = \angle ACB = \angle ACL$, whence $RNCL$ is cyclic. Similarly, $RMBL$ is cyclic.



Second soln. Denote by T the midpoint of MN. Since $\triangle ABC$ and $\triangle ANM$ are similar, with respective medians AO and AT, we have $\angle BAO = \angle CAT$. Thus the bisector AR of $\angle BAC$ also bisects $\angle OAT$. Therefore $\frac{RT}{RO} = \frac{AT}{AO}$. Furthermore, using the same fact again,

$$\frac{AT}{AO} = \frac{MN}{BC} = \frac{MT}{BO} = \frac{MT}{MO}.$$

We conclude that MR bisects $\angle OMN$. Now $\angle BMO = \angle B$ (O is the centre of the circle $(BCNM)$. Combined with $\angle AMN = \angle C$, this yields $\angle OMN = \angle A$, and hence $\angle BMR = \angle B + \angle A/2 = \angle CLR$. So B, L, R, M are concyclic. Likewise, C, L, R, N are concyclic.

2. Find all polynomials $P(x)$ with real coefficients which satisfy the equality

$$P(a-b)+P(b-c)+P(c-a)=2P(a+b+c)$$

for all triples a, b, c of real numbers such that $ab+bc+ca=0$.

Soln. Put $a=b=c=0$, we get $3P(0)=2P(0)$ which implies $P(0)=0$. Put $a=b=0$, we get $P(-c)=P(c)$ for all real c. Thus the polynomial has only even powers: $P(x)=\sum_{m=1}^{k} a_{2m}x^{2m}$.

Since the expression $ab+bc+ca=0$ is homogeneous, we may assume that $a=1$. This yields $b=-c/(1+c)$. Thus a general solution is

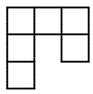
$$a=1+t, \quad b=-t, \quad c=t+t^2.$$

Let

$$\begin{aligned} A(n) &= (a-b)^n+(b-c)^n+(c-a)^n=(1+2t)^n+(t^2+2t)^n+(t^2-1)^n \\ B(n) &= 2(a+b+c)^n=2(t^2+t+1)^n \end{aligned}.$$

It is easily seen that by direct computation that $A(2)=B(2)$ and $A(4)=B(4)$. Now consider $n \geq 6$. Again direct computation shows that the coefficients of t^{2n} and t^{2n-1} of $A(n)$ and $B(n)$ are equal. The coefficient of t^{2n-2} of $A(n)$ is $4\binom{n}{2}-\binom{n}{1}$ while that of $B(n)$ is $2\left(\binom{n}{2}+\binom{n}{1}\right)$. It's clear that the latter is strictly smaller than the former for $n \geq 6$. This shows that $P(a-b)+P(b-c)+P(c-a)$ and $2P(a+b+c)$, as polynomials in t, are not identical if $k \geq 3$ (the terms in t^{4k-2} are different) and are identical if $k \leq 2$. So the answer is $P(x)=ax^2+bx^4$, where a, b are real numbers.

3. Define a *hook* to be a figure made up of six unit squares as shown in the diagram



or any of the figures obtained by applying rotations and reflections to this figure.

Determine all $m \times n$ rectangles that can be covered with hooks so that

(i) the rectangle is covered without gaps and without overlaps;

(ii) no part of a hook covers area outside the rectangle.

Soln. For any hook A, there is unique hook B covering the "inside" square of A with one of its "endmost" squares. In turn, the "inside" square of B must be covered by an "endmost" square of A. Thus, the hooks must come in pairs, in two different ways as shown in the figure below. We call such a pair a tile.

Depending on how the tile is placed, there are two cases:

(a) The column lengths of the tile are odd (3) while the row lengths are even (2 or 4) as shown in the figure above.

(b) The column lengths of the tile are even while the row lengths are odd.

Suppose a tiling a possible, then $12 \mid mn$. Also it is easy to see that $m, n \neq 1, 2, 5$. We shall prove that one of m, n is a multiple of 4. If the number of tiles is even, then $24 \mid mn$ and thus one of m, n is divisible by 4 and we are done. Thus we may assume that the number of tiles is odd. So one of (a) and (b) must occur an odd number of times. By symmetry, we may assume that (a) occurs an odd number of times.

Colour black every 4th column from the left. Each type (a) tile must intersect one black column. Since each type (b) tile can cover an even number of black squares, we see that the total number of black squares is odd. Thus the column length is odd, i.e, one of m, n is odd and so the other is divisible by 4.

Conversely, suppose $12 \mid mn$, 4 divides one of m, n, and none of the sides is 1,2, or 5. If $4 \mid m$ and $3 \mid n$, then we can easily cover the rectangle with the 3×4 tiles.

If $12 \mid m$ and $3 \nmid n$, then write $n = 3p + 4r$. We can then partition the rectangle into $m \times 3$ and $m \times 4$ rectangles. So a covering is again possible.

4. Let $n \geq 3$ be an integer. Let $t_1, t_2, \ldots, t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

Soln. By symmetry if suffices to show that $t_1 < t_2 + t_3$. We have

$$\begin{aligned} \text{RHS } &= n + \sum_{1 \leq i < j \leq n} \frac{t_i}{t_j} + \frac{t_j}{t_i} \\ &= n + t_1\left(\frac{1}{t_2} + \frac{1}{t_2}\right) + \frac{1}{t_1}(t_2 + t_3) + \sum_{\substack{(i,j) \neq (1,2),(1,3) \\ 1 \leq i < j \leq n}} \left(\frac{t_i}{t_j} + \frac{t_j}{t_i}\right) \end{aligned}$$

By AM-GM,

$$\frac{1}{t_2} + \frac{1}{t_2} \geq \frac{2}{\sqrt{t_2 t_3}}, \quad t_2 + t_3 \geq 2\sqrt{t_2 t_3}, \frac{t_i}{t_j} + \frac{t_j}{t_i} \geq 2.$$

Thus, setting $x = t_1/\sqrt{t_2 t_3} > 0$, we get

$$n^2 + 1 > n + 2x + \frac{2}{x} + 2\left[\binom{n}{2} - 2\right] = 2x + \frac{2}{x} + n^2 - 4.$$

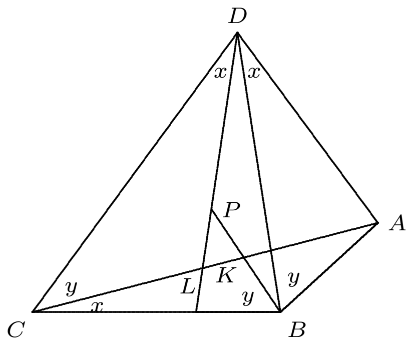
Hence $2x^2 - 5x + 2 < 0$, which implies that $1/2 < x < 2$. Therefore $t_1 < 2\sqrt{t_2 t_3} \leq t_2 + t_3$ as required.

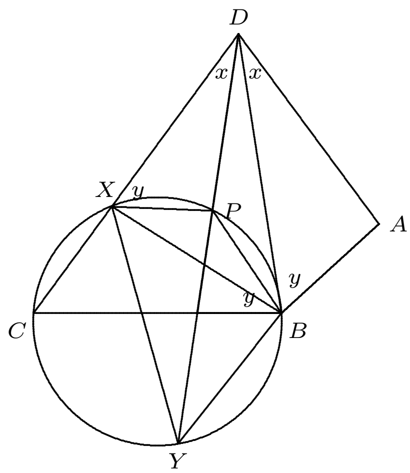
5. In a convex quadrilateral $ABCD$ the diagonal BD bisects neither the angle ABC nor the angle CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \qquad \text{and} \qquad \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

Soln.





Since P is in the interior of $ABCD$, we have $\angle DBA < \angle DBC$ iff $\angle BDA < \angle BDC$. So we may assume that P lies in triangles ACD and BCD. Also let $\angle PBC = \angle DBA = y$ and $\angle PDC = \angle BDA = x$.

Assume that $ABCD$ is cyclic. Let the lines BP and DP meet AC at K and L, respectively. Then $\angle ACB = \angle ADB = x$, $\angle ABD = \angle ACD = y$. Therefore $\angle PLK = x + y = \angle PKL$, whence $PK = PL$. Also $\triangle ADL \simeq \triangle BDC$ and $\triangle ABD \simeq \triangle KBC$. Hence

$$\frac{AL}{BC} = \frac{AD}{BD} = \frac{KC}{BC},$$

yielding $AL = KC$. Combine with the conclusions above, this implies that $\triangle ALP \equiv \triangle CKP$. Hence $AP = CP$.

Conversely, assume that $AP = CP$. Let the circumcircle of BCP meet the lines CD and DP again at X and Y, respectively. The $\triangle ADB \simeq \triangle PDX$. Therefore $DA/DP = DB/DX$ and so $\triangle ADP \simeq \triangle BDX$. Therefore

$$\frac{BX}{AP} = \frac{BD}{AD} = \frac{XD}{PD}.$$

Moreover, $\triangle DPC \simeq \triangle DXY$, which gives $\frac{YX}{CP} = \frac{XD}{PD}$. Since $AP = CP$, we have $BX = YX$. Hence

$$\angle DCB = \angle XYB = \angle XBY = \angle XPY = x + y = 180^\circ - \angle BAD.$$

This implies that $ABCD$ is cyclic.

6. We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.

Find all positive integers n such that n has a multiple which is alternating.

Soln. (Official) A positive integer is alternating iff it is not a multiple of 20.

The last two digits of a multiple of 20 are both even and so it can't be alternating.

For the other integers, there are several cases.

a) Every power of 2 has an alternating multiple with an even number of digits.

Proof: If suffices to construct an infinite sequence $\{a_n\}$ such that

$$a_n \equiv n+1 \pmod 2, \quad 2^{2n-1} \parallel \overline{a_{2n-1}a_{2n-2}\dots a_1}; \quad 2^{2n+1} \parallel \overline{a_{2n}a_{2n-1}\dots a_1}$$

for each n. (Here for any positive integers a, b, c, $a^b \parallel c$ means b is the largest integer such that $a^b \mid c$.) Start with $a_1 = 2$, $a_2 = 7$. If the sequence is constructed up to a_{2n}, set $a_{2n+1} = 4$. Then a_{2n+1} is even, and

$$2^{2n+1} \parallel \overline{a_{2n+1}\dots a_1} = 4 \cdot 10^{2n} + \overline{a_{2n}\dots a_1},$$

because $2^{2n+1} \parallel \overline{a_{2n}\dots a_1}$ by the induction hypothesis and $2^{2n+2} \parallel 4 \cdot 10^{2n}$. Denote $\overline{a_{2n+1}\dots a_1} = 2^{2n+1}A$, with A odd. Now a_{2n+2} must be odd and such that

$$2^{2n+3} \parallel \overline{a_{2n+2}\dots a_1} = a_{2n+2}10^{2n+1} + \overline{a_{2n+1}\dots a_1} = 2^{2n+1}[a_{2n+2}5^{2n+1} + A],$$

which holds whenever $5a_{2n+2} + A \equiv 4 \pmod 8$. The solutions of the last congruence are odd, since A is odd. In addition, a solution a_{2n+2} can be chosen from $\{0, \dots, 7\}$. The construction is complete.

b) Each number of the form $2 \cdot 5^n$ has an alternating multiple with an even number of digits.

Proof: We construct an infinite sequence $\{b_n\}$ such that

$$b_n \equiv n+1 \pmod 2 \quad \text{and} \quad 2 \cdot 5^n \mid \overline{b_n \dots b_1}$$

for each n. Start with $b_1 = 0$ and $b_2 = 5$. Suppose $b_1, \dots, b_n$ have been constructed. Let $\overline{b_n \dots b_1} = 5^\ell B$, where $\ell \geq n$ and $5 \nmid B$. The next digit b_{n+1} must be such that $b_{n+1} \equiv n+2 \pmod 2$ and 5^{n+1} divides

$$\overline{b_{n+1}\dots b_1} = b_{n+1}10^n + \overline{b_n \dots b_1} = 5^n[b_{n+1}2^n + 5^{\ell-n}B].$$

The latter is true whenever $b_{n+1}2^n + B$ is divisible by 5. Now the system of simultaneous congruence $b_{n+1} \equiv n+2 \pmod 2$, $b_{n+1}2^n + B \equiv 0 \pmod 5$ has a solution by Chinese remainder theorem, since 2^n and 5 are coprime. Also, a solution b_{n+1} can be chosen in $\{0, \dots, 9\}$, as needed.

For the general case $n = 2^\alpha 5^\beta k$, where k is not a multiple of 5 or 2 and $\alpha \leq 1$. First we note that $2^\alpha 5^\beta$ has an alternating multiple M with an even number, say $2m$, of digits.

Thus all numbers of the form $\overline{MM\ldots M}$ are also alternating. We claim that one of them is a multiple of n. Consider the numbers

$$C_\ell = 1 + 10^{2m} + \cdots + 10^{2m(\ell-1)}, \quad \ell = 1, 2, \ldots, k+1$$

There exist $1 \le i < j \le k+1$ such that $C_i \equiv C_j \pmod{k}$. Hence $k \mid C_j - C_i = C_{j-i}10^{2mi}$. Since 10 is coprime to k, it follows that $k \mid C_{j-i}$. Now it is straightforward that $C_{j-i} \times M$, a number of the form $\overline{MM\ldots M}$ is an alternating multiple of n.

Soln. Solution by Joel Tay, (RJC). The answer is every positive integer n which is not a multiple of 20. If 20 divides n, then the last 2 digits of any multiple of n are even, hence cannot be alternating. If $20 \nmid n$, then either (1) $2 \,\|\, n$, (2) $2 \nmid n$ or (3) $2^2 \mid n, 5 \nmid n$. We consider these 3 cases separately.

Case (1): $2 \,\|\, n$. We can reduce this to case (2) as follow. In this case, $n/2$ is odd. By case (2), an alternating multiple of $n/2$ is obtained. If this number is even, then it is an alternating multiple of n. If it is odd, multiplying by 10 (that is adding a zero to its unit place) gives an alternating multiple of n.

Case (2a): $2 \nmid n$ and $5 \nmid n$. That is $(n, 10) = 1$. Consider the number $x = 1212\cdots12$, where the digits '12' are repeated k times. Here k is taking to be large, say larger than $n\phi(n)$, where ϕ is the Euler function. Since $(2, n) = 1$, there exists an integer y, $1 \le y \le n$ such that $2y \equiv -x \pmod{n}$. On the other hand, $(10, n) = 1$ implies that $10^{\phi(n)} \equiv 1 \pmod{n}$. Thus $2 \times 10^{\phi(n)} \equiv 2 \pmod{n}$ and $2 \times 10^{m\phi(n)} \equiv 2 \pmod{n}$ for all integer m. Therefore,

$$a = 2 + 2\times 10^{\phi(n)} + 2\times 10^{2\phi(n)} + \cdots + 2\times 10^{(y-1)\phi(n)} \equiv 2 + 2 + \cdots + 2 \equiv 2y \equiv -x \pmod{n}.$$

That is $a + x$ is a multiple of n. Note that x is alternating and the digits of a are all even and the length of a is shorter than the length of x. Thus $a + x$ is alternating.

Case (2b): $2 \nmid n$ and $5^c \,\|\, n$, where c is a positive integer. We shall first find an alternating multiple of 5^c having at most c digits. To do so, we construct inductively an alternating multiple $\overline{a_{m-1}\cdots a_2a_1a_0}$ of 5^m for $m = 1, 2, \ldots, c$. Take $a_0 = 5$, $a_1 = 2$. Suppose 5^m divides the alternating number $\overline{a_{m-1}\cdots a_2a_1a_0}$.

First note that 0×10^m, 2×10^m, 4×10^m, 6×10^m, 8×10^m are distinct mod 5^{m+1} and 5^m divides all of them. Similarly, 1×10^m, 3×10^m, 5×10^m, 7×10^m, 9×10^m are also distinct mod 5^{m+1} and 5^m divides all of them. So there exists exactly one number among 0×10^m, 2×10^m, 4×10^m, 6×10^m, 8×10^m or exactly one number among 1×10^m, 3×10^m, 5×10^m, 7×10^m, 9×10^m where we denote it by $a_m \times 10^m$ such that it is the additive inverse of $\overline{a_{m-1}\cdots a_2a_1a_0} \pmod{5^{m+1}}$. In other words, $\overline{a_ma_{m-1}\cdots a_2a_1a_0}$ is divisible by 5^{m+1}. Furthermore the parity of a_m can be chosen to be different from that of a_{m-1}, hence ensuring $\overline{a_ma_{m-1}\cdots a_2a_1a_0}$ is alternating. This completes the proof that there is an alternating multiple $\overline{a_{c-1}\cdots a_2a_1a_0}$ of 5^c having at most c digits.

Now consider the integer n. First choose a positive integer q such that $q\phi(n/5^c) > c$. Note that 5^c divides any number ending with $\overline{a_{c-1}\cdots a_2a_1a_0}$. As such consider the following number.

$$x = \begin{cases} \overline{101010\cdots 10a_{c-1}\cdots a_2a_1a_0} & \text{if } a_{c-1} \text{ is odd} \\ \overline{010101\cdots 01a_{c-1}\cdots a_2a_1a_0} & \text{if } a_{c-1} \text{ is even} \end{cases}$$

Here there are k '01's or '10's in x. Also k is chosen to be sufficiently large, say larger than $(q+n)\phi(n/5^c)$.

As noted before, 5^c divides x and x is alternating. Since $(2, n/5^c) = 1$, there exists an integer y with $1 \le y \le n/5^c$ such that $2y \equiv -x \pmod{n/5^c}$.

Also $2 \times 10^{m\phi(n/5^c)} \equiv 2 \pmod{n/5^c}$ for all integers m. Thus

$$a = 2\times 10^{q\phi(n/5^c)} + 2\times 10^{(q+1)\phi(n/5^c)} + \cdots + 2\times 10^{(q+y-1)\phi(n/5^c)} \equiv 2+2+\cdots 2 \equiv 2y \equiv -x$$

$\pmod{n/5^c}$. In decimal representation, $a = \overline{200\cdots 0200\cdots 0\cdots 200\cdots 0}$, where each block of '$200\cdots 0$' has at least c zeros since $q\phi(n/5^c) > c$. Thus $a+x$ is alternating as the addition of a to x does not affect the first c digits of x, all digits of a are even and x is long enough to have more digits than a. Therefore $a+x$ is alternating and divisible by $n/5^c$. Since $a+x$ is also divisible by 5^c and $(5^2, n/5^c) = 1$, it is divisible by n.

Case (3): $2^c \,\|\, n$, where $c \ge 2$ and $5 \nmid n$. We first construct an alternating multiple of 2^c. Take $x = \overline{1010\cdots 10}$ or $\overline{0101\cdots 010}$, so that it has c digits. Note that $2 \,\|\, x$. (That is $x/2$ is odd.) Now we use induction to construct an alternating multiple of 2^m $(2 \le m \le c)$ of the form $x + \overline{a_{m-2}a_{m-3}\cdots a_0}$, where all digits $a_0, a_1, \ldots, a_{m-2}$ are even.

For $m = 2$, take $a_0 = 6$. Then 2^2 divides $\overline{101\cdots 16}$ or $\overline{010\cdots 016}$. Suppose 2^m divides $x + \overline{a_{m-2}a_{m-3}\cdots a_0}$. Note that $0\times 10^{m-1}$ and $2\times 10^{m-1}$ are distinct mod 2^{m+1} and that 2^m divides both of them. Exactly one of them is congruent to $-x - \overline{a_{m-2}a_{m-3}\cdots a_0}$ $\pmod{2^{m+1}}$. Denote that one by $a_{m-1}\times 10^{m-1}$. Then $x + \overline{a_{m-1}a_{m-2}\cdots a_0} = x + \overline{a_{m-2}a_{m-3}\cdots a_0} + a_{m-1}\times 10^{m-1}$ is divisible by 2^{m+1}. Since $a_0 = 6$ and $a_1, \ldots, a_{m-1}$ are either 0 or 2, the number $x + \overline{a_{m-1}a_{m-2}\cdots a_0}$ is an alternating multiple of 2^{m+1}. Consequently, we have constructed an alternating multiple of 2^c having at most c digits.

Now we return to the case $n = 2^c k$, where $(2,k) = 1$, $(5,k) = 1$ and $c \ge 2$. Denote the alternating multiple of 2^c obtained above by $\overline{b_{c-1}b_{c-2}\cdots b_0}$. Consider

$$x = \begin{cases} \overline{101010\cdots 10b_{c-1}\cdots b_2b_1b_0} & \text{if } b_{c-1} \text{ is odd} \\ \overline{010101\cdots 01b_{c-1}\cdots b_2b_1b_0} & \text{if } b_{c-1} \text{ is even} \end{cases}$$

where '10' or '01' is repeated k times and k is chosen sufficiently large, say larger $(q-n-1)\phi(n/2^c)$ with $q\phi(n/2^c) > c$. Note that x is alternating and is divisible by 2^c.

Let y be an integer with $1 \le y \le n/2^c$ such that $2y \equiv -x \pmod{n/2^c}$. As before,

$$a = 2\times 10^{q\phi(n/2^c)} + 2\times 10^{(q+1)\phi(n/2^c)} + \cdots + 2\times 10^{(2+y-1)\phi(n/5^c)} \equiv 2+2+\cdots 2 \equiv 2y \equiv -x$$

$\pmod{n/2^c}$. That is $a+x$ is divisible by $n/2^c$ and it is alternating. Since $q\phi(n/2^c) > c$, the first c digits of $a+x$ is $\overline{b_{c-1}b_{c-2}\ldots b_0}$ and thus $a+x$ is divisible by 2^c. Therefore it is an alternating multiple of n.

Solution by Andre Kueh, (RJC). Let $n = 2^a 5^b k$, where a and b are non-negative integers and $(k, 2) = 1, (b, 5) = 1$. Since $(10, k) = 1$, there exists a positive integer m such that $10^m \equiv 1 \pmod{k}$. We first prove the following lemma.

Lemma There exists an alternating number j with $2m$ digits and an odd digit at the unit place such that $(j, k) = 1$. Furthermore there exists a positive integer α such that $\alpha j \equiv 1 \pmod{k}$.

Proof Let $j = (10^{2m} - 1)/99 + 2\ell 10^{2m-1}$, $\ell = 1, 2, 3$. In other words, j is the $2m$-digit number $\overline{\eta 10101 \cdots 01}$, where $\eta = 2, 4, 6$. As $(10^{2m} - 1, 10^{2m-1}) = 1$, we have $((10^{2m} - 1)/99, 10^{2m-1}) = 1$ so that the common factors of $(10^{2m} - 1)/99$ and $2\ell 10^{2m-1}$ are factors of 2ℓ. Since $(10^{2m} - 1)/99$ is odd, the common factors of $(10^{2m} - 1)/99$ and $2\ell 10^{2m-1}$ is either 1 or 3. Thus the common factors of $(10^{2m} - 1)/99$ and $(10^{2m} - 1)/99 + 2\ell 10^{2m-1}$ is either 1 or 3. Consequently, $(10^{2m} - 1, (10^{2m} - 1)/99 + 2\ell 10^{2m-1})$ is a factor of $99 \times 3 = 3^3 \times 11$. Since there are 3 possible choices for ℓ, we may choose ℓ such that $j = (10^{2m} - 1)/99 + 2\ell 10^{2m-1}$ is not divisible by both 3 and 11. Now for this choice of ℓ, we have $(j, 10^{2m} - 1) = 1$. Thus $(j, k) = 1$ as k divides $10^m - 1$. Since $(j, k) = 1$, there exists α such that $\alpha j \equiv 1 \pmod{k}$. This completes the proof of the lemma.

Note that j has $2m$ digits and a finite number of concatenations of j is still an alternating number.

Case (1): $a = b = 0$. That is $n = k$. Let $x = \overline{jj \cdots j}$, where there are n copies of j. In other words, $x = j + j \times 10^{2m} + \cdots + j \times^{2m(n-1)}$. Since $n (= k)$ divides $2^m - 1$, we have $10^{2m} \equiv 1 \pmod{n}$. Thus $x \equiv j + j + \cdots + j = nj \equiv 0 \pmod{n}$. That is x is an alternating multiple of n.

Case (2): $a \geq 2, b = 1$. This impels that n is a multiple of 20. As the last two digits of any multiple of 20 are always even, n cannot have an alternating multiple.

Case (3): $a = 0, b \geq 1$. That is $n = 5^b k$. First we construct inductively an alternating multiple of 5^b having at least b digits whose unit digit is odd. When $b = 1$, take the number 5 itself. Suppose $\overline{a_{b-1} a_{b-2} \cdots a_0}$ is an alternating multiple of 5^b. Then $\overline{a_{b-1} a_{b-2} \cdots a_0} \equiv c \times 5^b \pmod{5^{b+1}}$. Note that $0 \times 10^b, 2 \times 10^b, 4 \times 10^b, 6 \times 10^b, 8 \times 10^b$ are distinct residue classes modulo 5^{b+1}, so do $1 \times 10^b, 3 \times 10^b, 5 \times 10^b, 7 \times 10^b, 9 \times 10^b$. We may pick an additive inverse $a_b \times 10^b$ of $c \times 5^b$ modulo 5^{b+1} such that a_b and a_{b-1} are of opposite parity. Then $\overline{a_b a_{b-1} \cdots a_0} \equiv a_b \times 10^b + c \times c \times 10^b \equiv 0 \pmod{10^{b+1}}$, giving an alternating multiple of 5^{b+1}.

Note that adding any arbitrary digits to the front of this alternating multiple of 5^b will still be a multiple of 5^b. Thus we may add random odd and even digits to this alternating multiple of b taking care to ensure that it remains alternating until we obtain an alternating multiple of 5^b having $2pm$ digits for some positive integer p. Let this alternating multiple of 5^b be y. Note that the leftmost digit of y is even. Now let $q \equiv -y \pmod{k}$, where $q \in \{1, 2, \ldots, k\}$. Consider the number $x = \overline{jj \cdots jjy}$. Here the number x has $q\alpha$ copies of j, where α is the number provided by the lemma. As $10^m \equiv 1 \pmod{k}$, we have $x \equiv q\alpha j + y \equiv 0 \pmod{k}$. Since x is divisible by 5^b, it is an alternating multiple of n.

Case (4): $a \geq 1, b = 0$. That is $n = 2^a k$. First we construct inductively an alternating multiple of 2^a having at least a digits. Let $2^a = \overline{a_{s-1} a_{s-2} \cdots a_1 a_0}$. The unit digit a_0 is always even. If a_1 is also even, we may add $\overline{a_{s-1} a_{s-2} \cdots a_1 a_0}$ to itself a number of times until a carry occurs in the 10th place. Then the digit at the 10th place of the resulting number is odd. In this way, we obtain a multiple of 2^a of the form $\overline{p_r p_{r-1} \cdots p_1 p_0}$ such that at least the first 2 right-most digits are of opposite parity. Now suppose p_μ is the first non-alternating digit in the number $\overline{p_r p_{r-1} \cdots p_1 p_0}$. We consider the following two cases.

Suppose both p_μ and $p_{\mu-1}$ are odd. We add a certain multiple of $\overline{a_s a_{s-1} \cdots a_1 a_0 00 \cdots 0}$ to $\overline{p_r p_{r-1} \cdots p_\mu p_{\mu-1} \cdots p_1 p_0}$, where there are $(\mu - 1)$ zeros in the first number, until a carry occurs at the 10^μth place. Then the resulting multiple of 2^a will be alternating starting from right to left up to the $(\mu + 1)$th digit.

Suppose both p_μ and $p_{\mu-1}$ are even. We add a certain multiple of $\overline{a_s a_{s-1} \cdots a_1 a_0 00 \cdots 0}$ to $\overline{p_r p_{r-1} \cdots p_\mu p_{\mu-1} \cdots p_1 p_0}$, where there are $(\mu - 1)$ zeros in the first number, until the first time no carry occurs at the 10^μth place. Again, the resulting multiple of 2^a will be alternating starting from right to left up to the $(\mu + 1)$th digit.

Continue the above procedure until the alternating multiple of 2^a so obtained has at least a digits. Then the number obtained by taking the first a digits of this multiple of 2^a is again an alternating multiple of 2^a. That is we discard all the digits after the 10^a place. Now as in case (3), we may add random digits to the front of this number until it has $2pm + 1$ digits for some positive integer p. Denote the resulting alternating multiple of 2^a by y. Note that both the first and last digits of y are even.

Since $(10, k) = 1 \pmod{k}$, there exists $t \in \{1, 2, \ldots, k\}$ such that $10t \equiv 1 \pmod{k}$. Also let $q \in \{1, 2, \ldots, k\}$ be such that $q \equiv -y \pmod{k}$.

Consider the number $x = \overline{jj \cdots jjy}$. Here the number x has $tq\alpha$ copies of j, where α is the number provided by the lemma. Note that x is alternating as the unit digit of j is odd. As $10^m \equiv 1 \pmod{k}$ so that $10^{2mr+1} \equiv 10 \pmod{k}$ for any positive integers r, we have $x \equiv 10tq\alpha j + y \equiv q + y \equiv 0 \pmod{k}$. Since x is divisible by 2^a, it is an alternating multiple of $n = 2^a k$.

Case (5): $a = 1, b \geq 1$. By case (3), there is an alternating multiple $\overline{a_r a_{r-1} \cdots a_0}$ of $n/10$, where a_0 is odd. Then $\overline{a_r a_{r-1} \cdots a_0 0}$ is an alternating multiple of n.

Concluding all, we have shown that a positive integer n is alternating if and only if it is not a multiple of 20.