37th International Mathematical Olympiad

India, July 1996.

1. Let $ABCD$ be a rectangular board with $|AB| = 20$, $|BC| = 12$. The board is divided into 20×12 unit squares. Let r be a given positive integer. A coin can be moved from one square to another if and only if the distance between the centres of the two squares is $\sqrt{r}$. The task is to find a sequence of moves taking the coin from the square which has A as vertex to the square which has B as a vertex.

(a) Show that the task cannot be done if r is divisible by 2 or 3.

(b) Prove that the task can be done if $r = 73$.

(c) Can the task be done when $r = 97$?

Official solution: Let

$$A = \{(i,j) : 1 \le i \le 19, 0 \le j \le 11\}.$$

Our task is to move from $(0,0)$ to $(19,0)$ via the points of A such each move has length $\sqrt{r}$. Thus if we move from (x,y) to $(x+a, y+b)$, then $a^2+b^2 = r$. Such a move is known as type (a,b). A point (x,y) in A is said to be reachable if we move to (x,y) from $(0,0)$.

(a) If r is even, then for each reachable point (x,y), $x+y$ must be even. Thus $(19,0)$ is not reachable.

If $3|r$, then $3|y$ and $3|y$ for each reachable point (x,y). So $(19,0)$ is not reachable.

(b) Consider $r = 73 = 8^2 + 3^2$. Let a be the number of moves of type $(8,3)$ minus the number of moves of type $(-8,-3)$, b be the corresponding number for $\pm(8,-3)$, c the corresponding number for $\pm(3,8)$ and d be the corresponding number for $\pm(3,-8)$. If we reach $(19,0)$, then

$$8(a+b) + 3(c+d) = 19, \quad 3(a-b) + 8(c-d) = 0.$$

A solution is $(a+b, c+d) = (2,1)$ and $(a-b, c-d) = (-8,3)$ or $a = -3, b = 5, c = 2, d = -1$. Try with 3 moves of type $(-8,-3)$, 5 of type $(8,-3)$, 2 of type $(3,8)$ and 1 of type $(-3,8)$, we get, by trial and error:

$$(0,0), (3,8), (11,5), (19,2), (16,10), (8,7), (04), (8,1), (11,9), (3,6), (11,3), (19,0).$$

(c) If $r = 97$, then since the only way to write 97 as a sum of two squares is $97 = 9^2+4^2$, the moves must be of the types $(\pm 9, \pm 4)$ and $(\pm 4, \pm 9)$. Let $B = \{(i,j) : 0 \le i \le 19, 4 \le j \le 7\}$ and $C = A - B$. Then it can be verified that moves of the type $(\pm 9, \pm 4)$ takes a point in B to a point in C and vice versa. A move of the type $(\pm 4, \pm 9)$ always takes a point B to another point in B.

To reach $(19,0)$, an odd number of moves of the type $(\pm 9, \pm 4)$ is required. Since both $(0,0)$ and $(19,0)$ are in C, we see that $(19,0)$ is not reachable.

2. Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incentres of triangles APB, APC respectively. Show that AP, BD and CE meet at a point.

3. Let $S = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. Find all functions on S and taking their values in S such tat

$$f(m + f(n)) = f(f(m)) + f(n) \quad \text{forall } m, n \in S.$$

4. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. Find the least possible value that can be taken by the minimum of these two squares.

Let $15a + 16b = r^2$, $16a - 15b = s^2$. Thus

$$r^4 + s^4 = (15^2 + 16^2)(a^2 + b^2) = 481(a^2 + b^2) = 13 \times 37(a^2 + b^2).$$

Note that by Fermat's Little theorem, $x^4 \equiv -1$ has no solution both in mod 13 and mod 37.

Taking mod 13, we have $r, s \equiv 0 \pmod{13}$. Similarly, $r, s \equiv 0 \pmod{37}$. Thus r, s are both multiples of 481. It is easy ti check that $r = s = 481$ is a solution, with $a = 481 \times 31$ and $b = 481$. Thus the answer is 481^2.

5. Let $ABCDEF$ be a convex hexagon such that AB is parallel to ED, BC is parallel to FE and CD is parallel to AF. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let p denote the perimeter of the hexagon. Prove that

$$R_A + R_B + R_C \geq \frac{p}{2}.$$

6. Let n, p, q be positive integers with $n > p + q$. Let $x_0, x_1, \ldots, x_n$ be integers satisfying the following conditions:

(a) $x_0 = x_n = 0$;

(b) for each integer i with $1 \leq i \leq n$,

$$\text{either } x_i - x_{i-1} = p \quad \text{or } x_i - x_{i-1} = -q.$$

Show that there exists a pair (i, j) of indices with $i < j$ and $(i, j) \neq (0, n)$ such that $x_i = x_j$.