39th International Mathematical Olympiad

Taiwan, July 1998.

1. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

Soln. If $ABCD$ is a cyclic quad, then it is easy to show that $\angle APB + \angle CPD = 180°$. From here one easily concludes that the two areas are equal.

For the converse we use coordinate geometry. Let P be the origin. Let the coordinates of A and B be $(-a, -b)$ and $(a, -b)$, respectively where a and b are both positive. Let the midpoint of CD be (c, d). Then, since P is in the interior, C is $(c, d) - t(-d, c) = (c + td, d - tc)$ and D is $(c, d) + t(-d, c) = (c - td, d + tc)$, where $t > 0$. (The vector CD is in the direction $(-d, c)$.) Without loss of generality, let $c^2 + d^2 = 1$. Then area of PCD and APB are t and ab, respectively. Thus $t = ab$. The fact that AC is perpendicular to BD implies that

$$(c - td - a, d + tc + b) \cdot (c + td + a, d - tc + b) = 0.$$

This simplifies to

$$(1 - a^2)(b^2 + 2bd + 1) = 0.$$

We have

$$PA = PB = a^2 + b^2, \quad PC = PD = t^2 + 1 = a^2b^2 + 1.$$

Thus $PA = PB = PC = PD = b^2 + 1$ when $a^2 = 1$, i.e., A, B, C, D are on a circle with centre at P.

We now consider the case $b^2 + 2bd + 1 = 0$. Consider this as a quadratic equation in b, the discriminant $4d^2 - 4 \geq 0$ if and only if $d^2 \geq 1$. But we know that $d^2 \leq 1$. Thus $d^2 = 1$ and consequently $b = \pm 1$ or $b^2 = 1$. Since $b > 0$, we actually have $b = 1$ and $d = -1$. Thus $c = 0$ whence $A = C$ and $B = D$, which is impossible.

Soln. (official): Let AC and BD meet at E. Assume by symmetry that P lies in ΔBEC and denote $\angle ABE = \phi$ and $\angle ACD = \psi$. The triangles ABP and CDP are isosceles. If M and N are the respective midpoints of their bases AB and CD, then $PM \perp AB$ and $PN \perp CD$. Note that M, N and P are not collinear due to the uniqueness of P.

Consider the median EM to the hypotenuse of the right triangle ABE. We have $\angle BEM = \phi$, $\angle AME = 2\phi$ and $\angle EMP = 90° - 2\phi$. Likewise, $\angle CEN = \psi$, $\angle DNE = \psi$ and $\angle ENP = 90° - 2\psi$. Hence $\angle MEN = 90° + \phi + \psi$ and a direct computation yields

$$\angle NPM = 360° - (\angle EMP + \angle MEN + \angle ENP) = 90° + \phi + \psi = \angle MEN.$$

It turns out that, whenever $AC \perp BD$, the quadrilateral $EMPN$ has a pair of equal opposite angles, the ones at E and P.

We now prove our claim. Since $AB = 2EM$ and $CD = 2EN$, we have $[ABP] = [CDP]$ if and only if $EM \cdot PM = EN \cdot PN$, or $EM/EN = PN/PM$. On account of $\angle MEN = \angle NPM$, the latter is equivalent to $\Delta EMN \sim \Delta PNM$. This holds if and only if $\angle EMN = \angle PNM$ and $\angle ENM = \angle PMN$, and these in turn mean that $EMPN$ is a parallelogram. But the opposite angles of $EMPN$ at E and P are always equal, as noted above. So it is a parallelogram if and only if $\angle EMP = \angle ENP$; that is, if $90^\circ - 2\phi = 90^\circ - 2\psi$. We thus obtain a condition equivalent to $\phi = \psi$, or to $ABCD$ being cyclic.

2. In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

Soln. Form a matrix where columns represent the contestants and the rows represents the judges. And we have a 1 when the judge "passes" the corresponding contestant and a 0 otherwise. A pair of entries in the same column are "good" if they are equal. Thus the number of good pairs in any two rows is at most k whence the total number of good pairs in the matrix is at most $\binom{b}{2}k = kb(b-1)/2$. In any column, if there are i zeroes, then the total number of good pairs is $\binom{i}{2} + \binom{j}{2}$, where $j = b - i$. Write $b = 2m + 1$ (since b is odd), we have

$$\binom{i}{2} + \binom{j}{2} - m^2 = (m-i)^2 + (m-i) = (m-j)^2 + (m-j) \geq 0$$

since either $m - i \geq 0$ or $m - j \geq 0$. Thus the total number of good pairs is at least $am^2 = a(b-1)^2/4$. Therefore

$$a(b-1)^2/4 \leq kb(b-1)/2$$

from which the result follows.

3. For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that

$$\frac{d(n^2)}{d(n)} = k$$

for some n.

Soln. Note that an integer q satisfies $d(n^2)/d(n) = q$ for some q if and only if q is of the form

$$\frac{(4k_1+1)(4k_2+1)\dots(4k_i+1)}{(2k_1+1)(2k_2+1)\dots(2k_i+1)} \qquad (*)$$

(This follows from the fact that if $q = p_1^{k_1} \cdots p_i^{k_i}$ is the prime decomposition of q, then $d(q) = (k_1 + 1) \cdots (k_i + 1)$.) Thus m is necessarily odd. Thus we need to show that every odd number can be expressed in the same way. Certainly 1 and 3 can be so expressed as $1 = 1/1$ and $3 = \frac{5}{3}\frac{9}{5}$. Let p be an odd integer. We assume that every odd integer less than p can be written in the form $(*)$. We have

$$p + 1 = 2^m(2k + 1)$$

for some positive integer m and nonnegative integer k. If $m = 1$, then $p = 4k + 1 = \frac{4k+1}{2k+1}(2k + 1)$. Since $2k + 1 < p$, by the induction hypothesis, it can be expresses in the form $(*)$ and hence so can p.

Now suppose that $m > 1$. We have

$$p(2^m - 1) = 2^{2m-1}k - 2^m k + 2^{2m-2} - 2^m + 1 = 2^m x + 1$$

and

$$\frac{2^m x + 1}{2^{m-1}x + 1} \frac{2^{m-1}x + 1}{2^{m-2}x + 1} \cdots \frac{4x + 1}{2x + 1} = \frac{p(2^m - 1)}{2x + 1} = \frac{p}{2k + 1}$$

since $2x + 1 = (2^{m-1} - 1)(2k + 1)$. Since the left hand side is of the form $(*)$ and $2k + 1$ can be written in that form by the induction hypothesis, we conclude that p can also be written in the same form.

(Note: The main idea is that it is easy to solve the case where $p \equiv 1 \pmod 4$. For $p \equiv 3 \pmod 4$, we try to multiply p with an odd integer so that $p(4k + 3) = 4\ell + 1$. By considering small values of p it was found that $2^m - 1$ as defined above works.)

4. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

Soln. Since $ab^2 + b + 7 | b(a^2b + a + b)$ and $a^2b^2 + ab + b^2 = a(ab^2 + b + 7) + (b^2 - 7a)$, we have either $b^2 - 7a = 0$ or $b^2 - 7a$ is a multiple of $ab^2 + b + 7$. The former implies that $b = 7t$ and $a = 7t^2$. Indeed these are solutions for all positive t.

For the second case, we note that $b^2 - 7a < ab^2 + b + 7$. Thus $b^2 - 7a < 0$. For $ab^2 + b + 7$ to divide $7a - b^2$, $b = 1, 2$. The case $b = 1$ requires that $7a - 1$ be divisible by $a + 8$. The quotients are less than 7. Testing each of the possibilities yields $a = 49, 11$. These are indeed solutions.

The case $b = 2$ requires that $7a - 4$ be divisible by $4a + 11$. The quotient has to be 1 and this is clearly impossible.

5. Let I be the incentre of triangle ABC. Let the incircle of ABC touch the sides BC, CA and AB at K, L and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that $\angle RIS$ is acute.

Soln. (Use coordinate geometry) Let I be the origin and the coordinates of B be $(0, a)$. Let the inradius be 1. Then the coordinates of M and K are $(r, s$ and $(-r, s)$ where $r = \sqrt{a^2 - 1}/a$ and $s = 1/a$. Let the coordinate of L be (p, q). Then we have $p^2 + q^2 = 1$. Let the coordinates of R and S be (x', a) and $(x", a)$. Then $x' = [r(a-q)+p(s-a)]/(s-q) =$

$m+n$ where $m=\sqrt{a^2-1}(a-q)/(1-aq)$ and $n=p(1-a^2)/(1-aq)$ and $x" = -m+n$. Let P be the mid point of SR. Then $\angle RIS$ is acute if and only if $OP > m$. Now $OP^2 = a^2+n^2 > m^2$ if and only if $(aq-1)^2 > 0$. Thus we are done. (Note: From the proof one can conclude that result still holds if one replaces the incircle by the excircle and the incentre by the corresponding excentre.

Second soln. (official): Let $\angle A = 2a$, $\angle B = 2b$ and $\angle C = 2c$. Then we have

$$\angle BMR = 90^\circ - a, \quad \angle MBR = 90^\circ - b, \quad \angle BRM = 90^\circ - c.$$

Hence $BR = BM\cos a/\cos c$. Similarly $BS = BK\cos c/\cos a = BL\cos a/\cos a$. Thus

$$\begin{aligned} IR^2 + IS^2 - RS^2 &= (BI^2+BR^2)+(BI^2+BS^2)-(BR+BS)^2 \\ &= 2(BI^2 - BR\cdot BS) = 2(BI^2-BK^2) = 2IK^2 > 0 \end{aligned}$$

So by the cosine law, $\angle RIS$ is acute.

6. Consider all functions f from the set $\mathbb{N}$ of all positive integers into itself satisfying

$$f\left(t^2 f(s)\right) = s\left(f(t)\right)^2,$$

for all s and t in $\mathbb{N}$. Determine the least possible value of $f(1998)$.

Soln. *(Official solution)*: Let f be a function that satisfies the given conditions and let $f(1)=a$. By putting $s=1$ and then $t=1$, we have

$$f(at^2) = f(t)^2, \quad f(f(s)) = a^2 s. \quad \text{for all } s, t.$$

Thus

$$\begin{aligned} (f(s)f(t))^2 &= f(s)^2 f(at^2) = f(s^2 f(f(at^2))) \\ &= f(s^2a^2at^2) = f(a(ast)^2) \\ &= f(ast)^2 \end{aligned}$$

It follows that $f(ast) = f(s)f(t)$ for all s,t; in particular $f(as) = af(s)$ and so

$$af(st) = f(s)f(t) \quad \text{for all } s,t.$$

From this it follows by induction that

$$f(t)^k = a^{k-1}f(t^k), \quad \text{for all } t,k.$$

We next prove that $f(n)$ is divisible by a for each n. For each prime p, let p^α and p^β be highest power of p that divides a and $f(n)$, respectively. The highest power of p that divides $f(n)^k$ is $p^{k\beta}$ while that for a^{k-1} is $p^{(k-1)\alpha}$. Hence $k\beta \geq (k-1)\alpha$ for all k which is possible only if $\beta \geq \alpha$. Thus a divides $f(n)$.

Thus the new function $g(n) = f(n)/a$ satisfies

$$g(a) = a, \quad g(mn) = g(m)g(n), \quad g(g(m)) = m, \qquad \text{for all } m, n.$$

The last follows from

$$\begin{aligned} ag(g(m)) &= g(a)g(g(m)) = g(ag(m)) = g(f(m)) \\ &= f(f(m))/a = a^2m/m = am \end{aligned}$$

It is easy to show that g also satisfies all the conditions and $g(n) \leq f(n)$. Thus we can restrict our consideration to g.

Now g is an injection and takes a prime to a prime. Indeed, let p be a prime and let $g(p) = uv$. Then $p = g(g(p)) = g(uv) = g(u)g(v)$. Thus one of the factors, say $g(u) = 1$. Then $u = g(g(u)) = g(1) = 1$. Thus $g(p)$ is a prime. Moreover, $g(m) = g(n)$ implies that $m = g(g(m)) = g(g(n)) = n$.

To determine the minimum value, we have $g(1998) = g(2 \cdot 3^3 \cdot 37) = g(2)g(3)^3g(37)$. Thus a lower bound for $g(1998)$ is $2^3 \cdot 3 \cdot 5 = 120$. There is also a g with $g(1998) = 120$. This is obtained by defining $g(3) = 2, g(2) = 3, g(5) = 37, g(37) = 5$, and $g(p) = p$ for all other primes.