Singapore International Mathematical Olympiad
Training Problems

8 February 2003

1. Determine whether there exist an integer polynomial $f(x)$ together with integers a, b and c satisfying the following conditions.

   (i) $ac \neq bc$.

   (ii) $f(a) = a, f(b) = b, c^2 + f(c)^2 + f(0)^2 = 2cf(0)$.

2. Let $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c, d are real constants. Suppose $p(1) = 827, p(2) = 1654$ and $p(3) = 2481$. Find the value of $(p(9) + p(-5))/4$.

3. How many integer polynomials of the form $x^3 + ax^2 + bx + c = 0$ having a, b, c as roots are there?

1. Determine whether there exist an integer polynomial $f(x)$ together with integers a, b and c satisfying the following conditions.

(i) $ac \neq bc$.

(ii) $f(a) = a, f(b) = b, c^2 + f(c)^2 + f(0)^2 = 2cf(0)$.

Solution The second condition in (ii) implies that $(c - f(0))^2 + f(c)^2 = 0$. That is $f(c) = 0$ and $f(0) = c$. As f is an integer polynomial and $f(a) = a$, we have $(a - c)|(f(a) - f(c))$. That is $(a - c)|a$. Also $(a - 0)|(f(a) - f(0))$ so that $a|(a - c)$. By (i), $c \neq 0$. Thus, $a - c = -a$ giving $c = 2a$. Similarly, $c = 2b$. But this gives $a = b$, contradicting (i). Therefore, no such integer polynomial exists.

2. Let $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c, d are real constants. Suppose $p(1) = 827, p(2) = 1654$ and $p(3) = 2481$. Find the value of $(p(9) + p(-5))/4$.

Solution Let $q(x) = p(x) - 827x$. Then $q(x)$ is a polynomial of degree 4. As $q(1) = q(2) = q(3) = 0$, we have $q(x) = (x - 1)(x - 2)(x - 3)(x - r)$, for some r. Therefore, $\frac{1}{4}(p(9) + p(-5)) = \frac{1}{4}(q(9) + q(-5)) + 827 = \frac{1}{4}((8)(7)(6)(9 - r) + (6)(7)(8)(5 + r)) = 1176 + 827 = 2003$.

3. How many integer polynomials of the form $x^3 + ax^2 + bx + c = 0$ having a, b, c as roots are there?

Solution From the relation between roots and coefficients of a polynomial equation, we have

$$a + b + c = -a \quad (1)$$
$$ab + bc + ac = b \quad (2)$$
$$abc = -c \quad (3)$$

From (3), $c = 0$ or $ab = -1$.

Case 1. $c = 0$. Substituting this into (1), we obtain $b = -2a$. Using (2), $ab = b$. Thus, $a = 1, b = -2$ or $a = b = 0$.

Case 2. $c \neq 0$ and $ab = -1$. As a and b are integers, we must have $a = 1, b = -1$ or $a = -1, b = 1$. When $a = 1, b = -1$, we get $c = -1$ using (1). These values of a, b, c also satisfy (2). When $a = -1, b = 1$, we get $c = 1$. But then these values of a, b, c do not satisfy (2).

Therefore, there are altogether 3 such polynomials.