Singapore International Mathematical Olympiad
Training Problems

18 January 2003

1. Let M be a point on the segment AB. Squares $AMCD$ and $MBEF$ are erected on the same side of AB with F lying on MC. The circumcircles of $AMCD$ and $MBEF$ meet at a second point N. Prove that N is the intersection of the lines AF and BC.

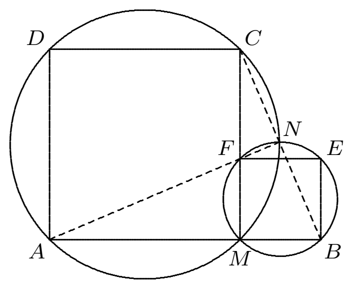
2. Let DM be the diameter of the incircle of a triangle ABC where D is the point at which the incircle touches the side AC. The extension of BM meets AC at K. Prove that $AK = CD$.

3. Tangents PA and PB are drawn from a point P outside a circle Γ. A line through P intersects AB at S and Γ at Q and R. Prove that PS is the harmonic mean of PR and PQ.

4. (IMO 1981)Three circles of equal radius have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point O.

1. Let M be a point on the segment AB. Squares $AMCD$ and $MBEF$ are erected on the same side of AB with F lying on MC. The circumcircles of $AMCD$ and $MBEF$ meet at a second point N. Prove that N is the intersection of the lines AF and BC.

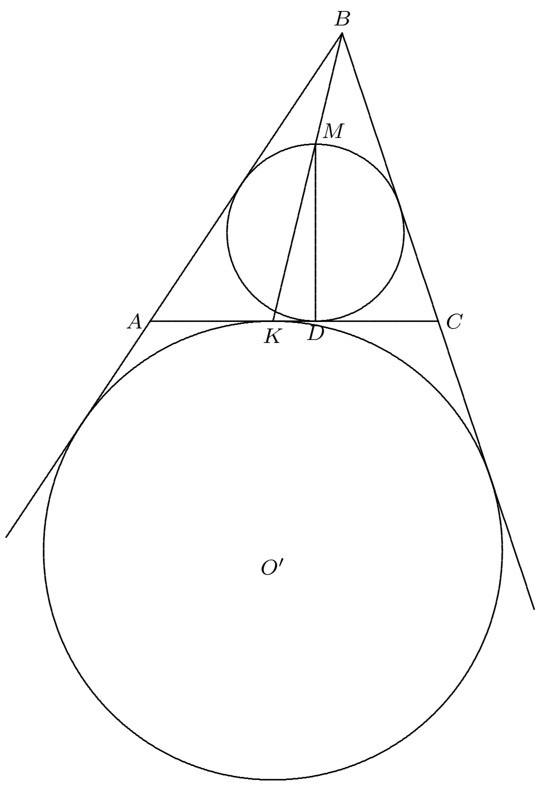
Solution



Let AF intersect BC at N'. We wish to show that $N = N'$. As $\triangle AMF$ is congruent to $\triangle CMB$, we have $\angle AN'C = 90°$ so that N' lies on the circle with AC as diameter. That is N' lies on the circumcircle of $AMCD$. Similarly, N' lies on the circumcircle of $MBEF$. Thus $N = N'$.

2. Let DM be the diameter of the incircle of a triangle ABC where D is the point at which the incircle touches the side AC. The extension of BM meets AC at K. Prove that $AK = CD$.

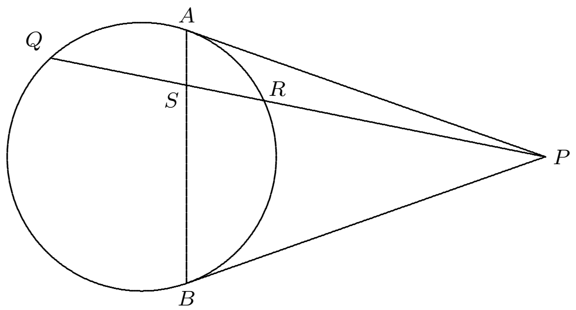
Solution



Consider a homothety centered at B carrying the incircle to the excircle. The diameter MD of the incircle is mapped to the diameter $M'D'$ of the excircle. Since MD is perpendicular to AC, $M'D'$ is also perpendicular to AC. Therefore M' must be the point K. That is the excircle touches AC at K. Therefore, $AK = (a+b-c)/2 = CD$.

3. Tangents PA and PB are drawn from a point P outside a circle Γ. A line through P intersects AB at S and Γ at Q and R. Prove that PS is the harmonic mean of PR and PQ.

Solution Since $\triangle APQ$ is similar to $\triangle RPA$, we have $PQ/PA = AQ/AR$. Also $\triangle BPQ$ is similar to $\triangle RPB$, we have $PR/PB = RB/QB$. Dividing the second equation by the first equation and using the fact that $PA = PB$, we obtain $PR/PQ = (RB/AQ) \cdot (AR/QB) = (RS/AS) \cdot (AS/QS) = SR/SQ$. This shows that the ratio that S divides QR internally is the same as the ratio that P divides QR externally. This determines the position of S on the segment QR.



Thus

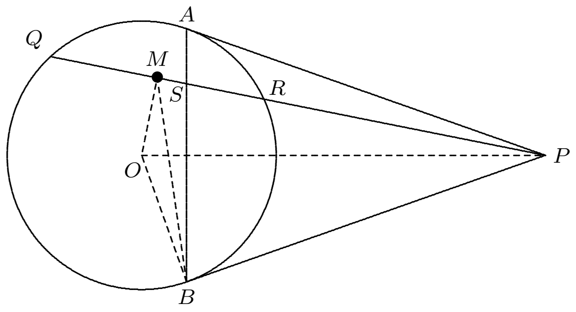
$$SR = QR \cdot \frac{PR}{PR+PQ}, \quad SQ = QR \cdot \frac{PQ}{PR+PQ}.$$

Also

$$PS = PR + RS = PR + \frac{QR \cdot PR}{PR+PQ} = PR + \frac{(PQ-PR) \cdot PR}{PR+PQ} = \frac{2PR \cdot PQ}{PR+PQ}.$$

That is PS is the harmonic mean of PR and PQ.

(Second Solution by Colin Tan) Let M be the midpoint of QR. Then to prove $PS = 2PR\cdot PQ/(PR+PQ)$ is equivalent to prove that $PS\cdot PM = PR\cdot PQ$. Or equivalently, $PS\cdot PM = PB^2$, since $PR\cdot PQ = PB^2$. Therefore, we have to show that PB is tangent to the circumcircle of $\triangle SMB$. Let O be the centre of Γ. Then O, M, P, B are concyclic and OP is perpendicular to AB. Hence, $\angle PBA = \angle POB = \angle SMB$. Therefore, PB is tangent to the circumcircle of $\triangle SMB$.



(Third Solution) Applying Stewart's Theorem to $\triangle APB$, we have

$$(AS + SB) \cdot PS^2 + (AS + SB) \cdot AS \cdot SB = AP^2 \cdot SB + BP^2 \cdot AS.$$

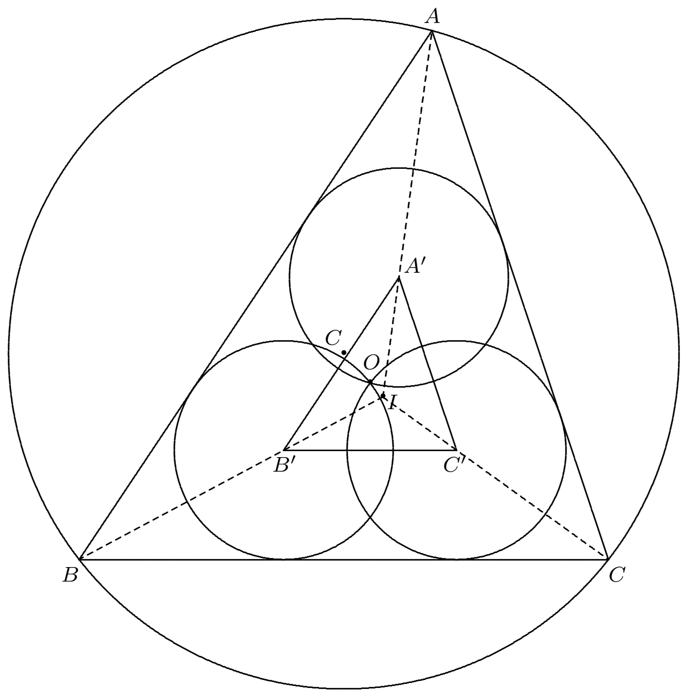
Since $AP = BP$, we may cancel the common factor $(AS + SB)$, thus obtaining

$$PS^2 + AS \cdot SB - AP^2 = 0.$$

Since $AS \cdot SB = QS \cdot SR = (PQ - PS)(PS - PR) = (PQ + PR) \cdot PS - PS^2 - PQ \cdot PR$ and $PA^2 = PQ \cdot PS$, we have $(PQ + PR) \cdot PS = 2PQ \cdot PR$. Thus, PS is the harmonic mean of PR and PQ.

4. (IMO 1981)Three circles of equal radius have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point O.

Solution



Let A', B', C' be the centres of the circles inside $\triangle ABC$. As AA', BB', CC' are angle bisectors, they meet at the incenter I of triangle ABC. I is also the incenter of the triangle $A'B'C'$. The circles are of the same radii. Thus A' and B' are of equal distance from AB so that AB is parallel to $A'B'$. Similarly, BC is parallel to $B'C'$ and $A'C'$ is parallel to AC. That is $\triangle ABC$ is similar to $\triangle A'B'C'$. Consider a homothety centred at I sending A' to A, B' to B and C' to C. Thus the circumcentre O of $\triangle A'B'C'$ is mapped to the circumcentre C of $\triangle ABC$ under this homothety. Therefore, I, C, O are collinear.